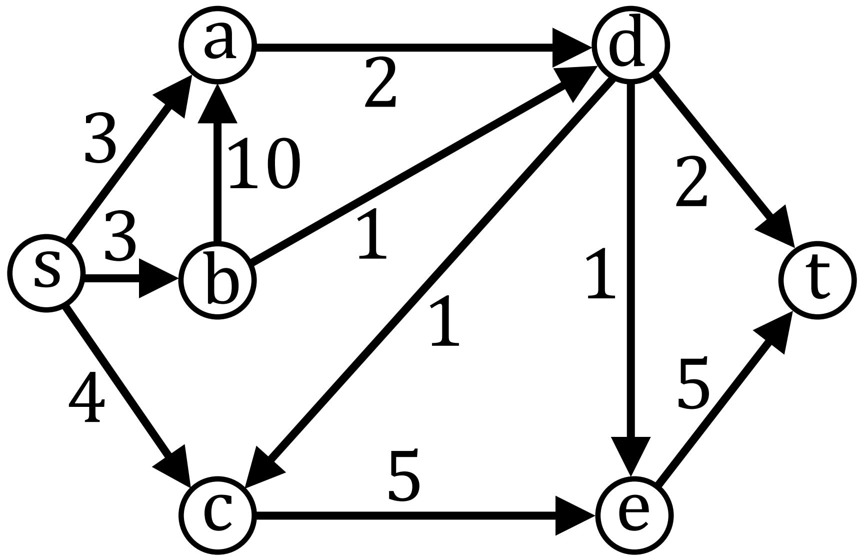
Flow Networks
CSCI 532

Motivation

Suppose we have a directed graph that represent an oil pipeline network. Edge weight represent pipe capacity. How much oil can we transfer from source s to sink t?



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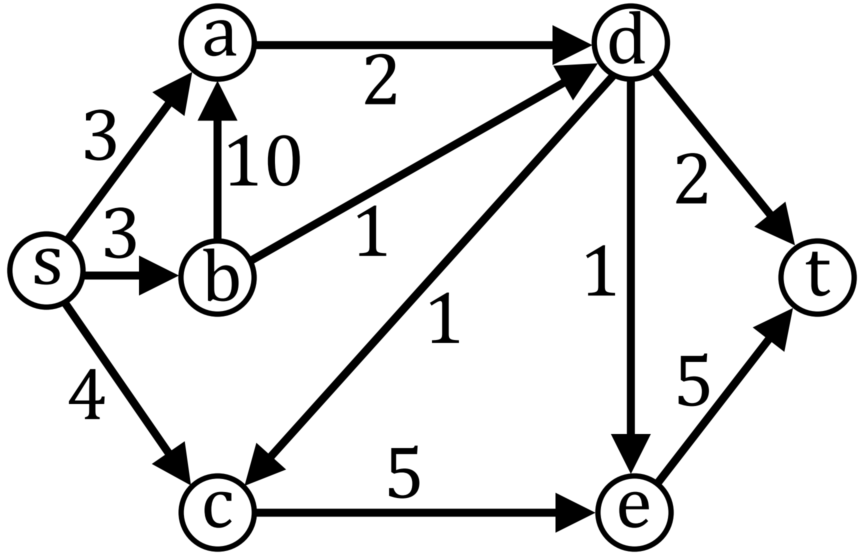

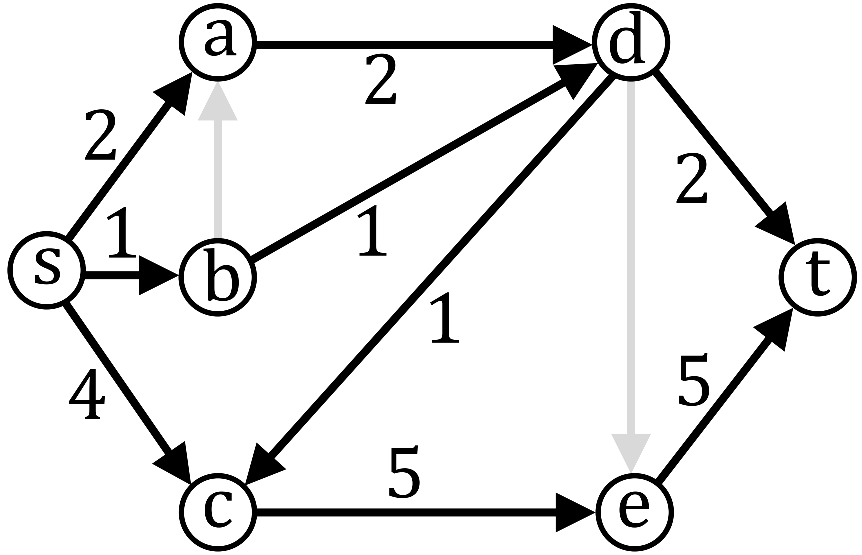

Total flow = 7

Flow Network

Flow Network:

An $s - t$ flow is a function $f \colon E \to \mathbb{R}^+$ such that:

Flow Network

Flow Network:

- Directed-edge graph, $G = (V, E)$.
- Finite positive edge capacity, c_e.
- Single source, s, without input edges.
- Single sink, t, without output edges.

We'll also sometimes use the assumption that the capacities are positive integer values.

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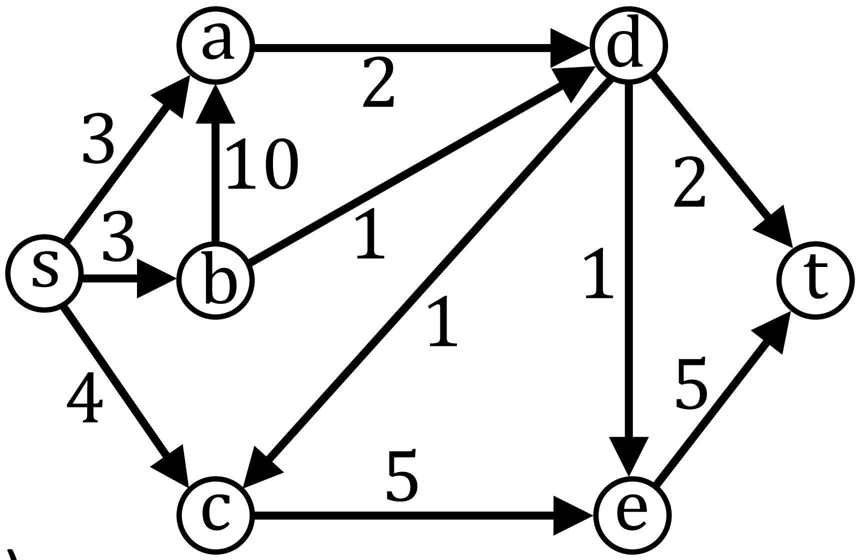

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- $0 \leq f(e) \leq c_e, \forall e \in E$. (capacity constraint)

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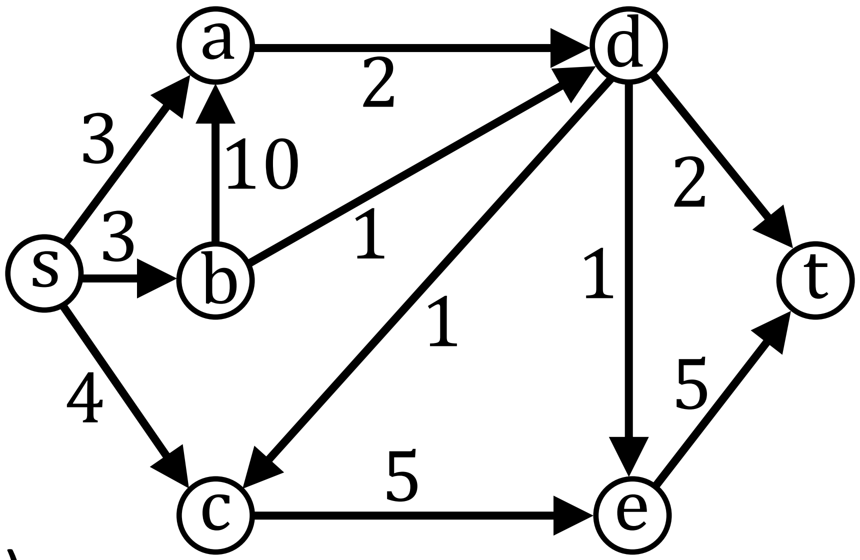

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- $0 \leq f(e) \leq c_e, \forall e \in E$. (capacity constraint)
- $\sum_{e \epsilon \text{input}(v)} f(e) = \sum_{e \in \text{output}(v)} f(e), \forall v \in V \setminus \{s, t\}$.
  (conservation of flow constraint: “Everything that goes into a node has to come out, except for s and t”)

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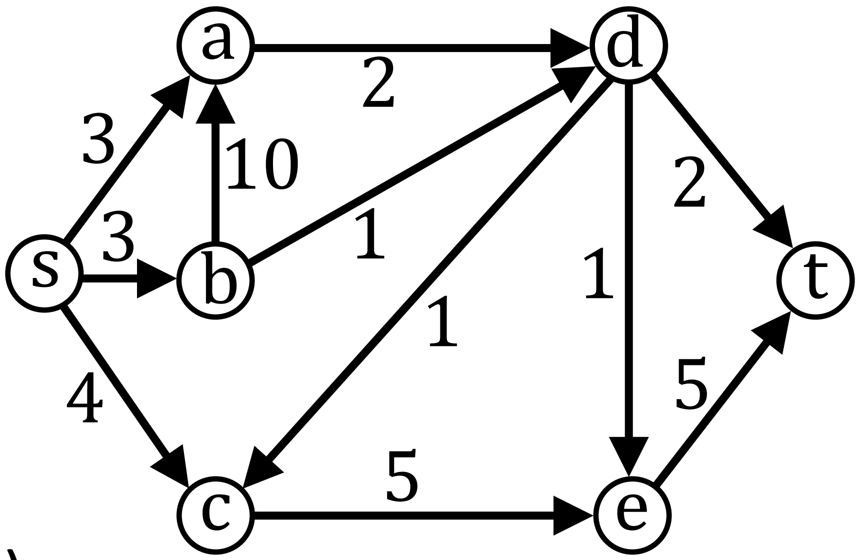

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- Value of flow = $val(f)$ = $\sum_{e \epsilon \text{output}(s)} f(e)$ = $\sum_{e \epsilon \text{input}(t)} f(e)$

Flow Network

Maximum Flow Problem:
Given a flow network, find the maximum possible value of flow.

Flow Network:

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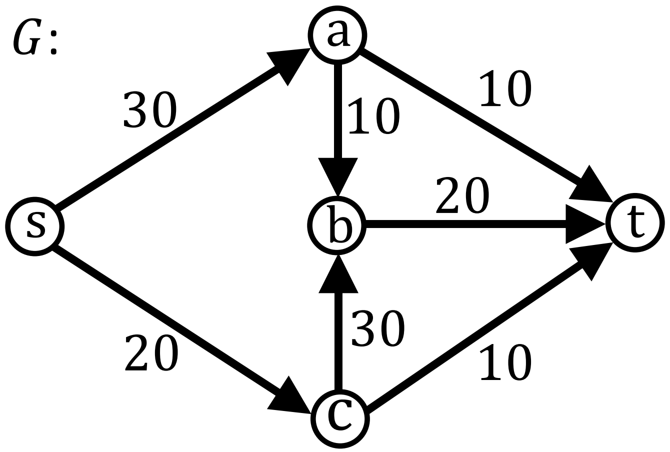
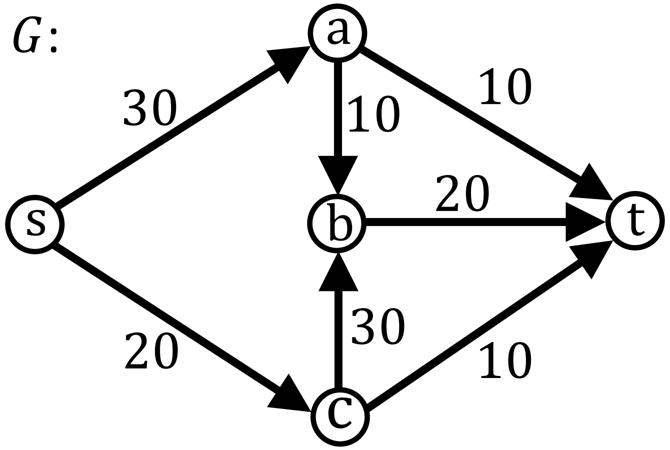
Max Flow Algorithm



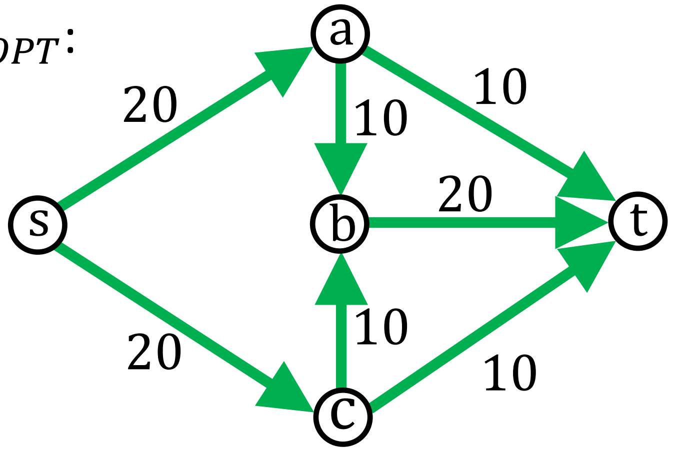
Max Flow?

Max Flow Algorithm

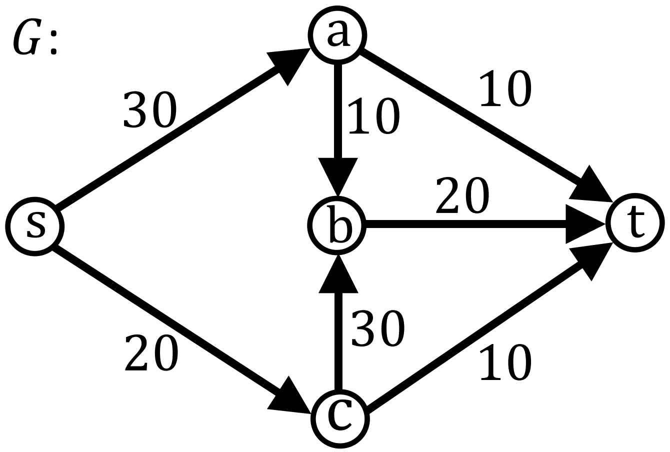
G:

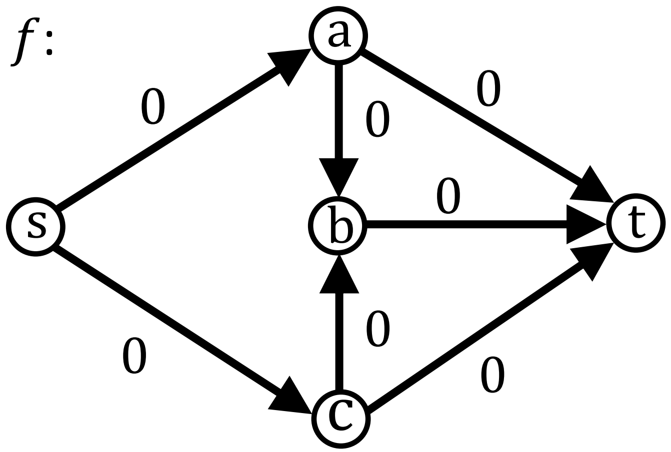


f_{OPT}:

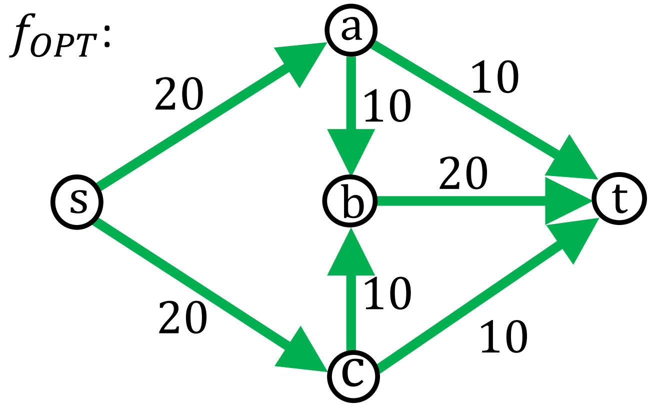


Max Flow Algorithm

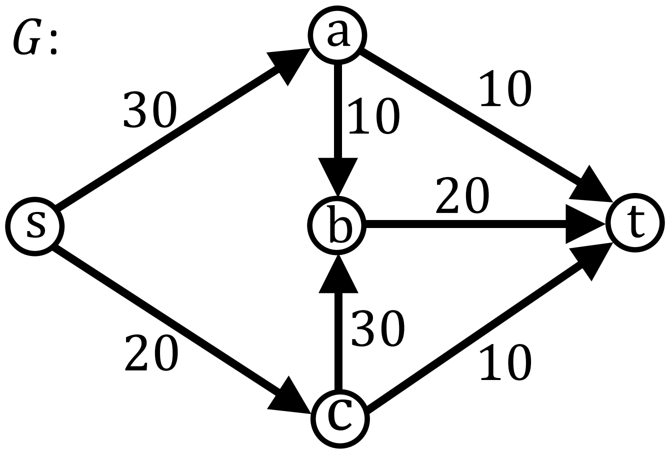


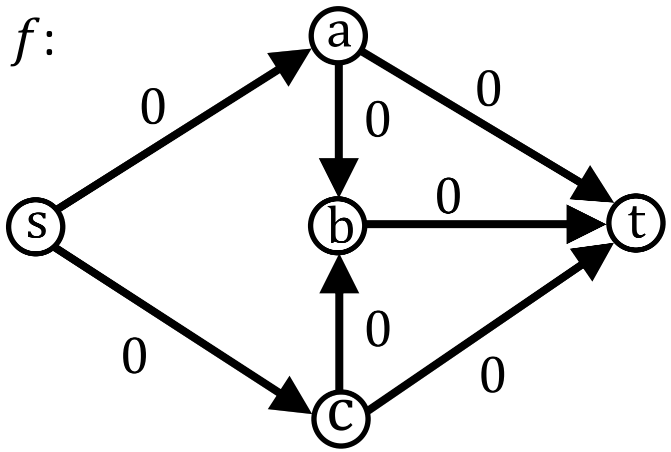


Ideas?



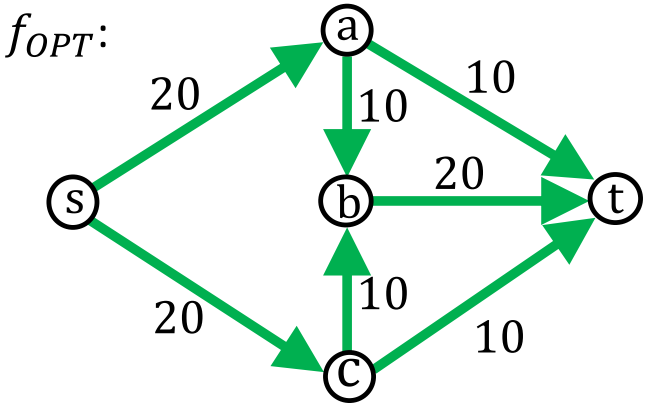
Max Flow Algorithm





Ideas?

Somehow, we are going to have to put flow on edges. Should we select edges or paths?



Max Flow Algorithm

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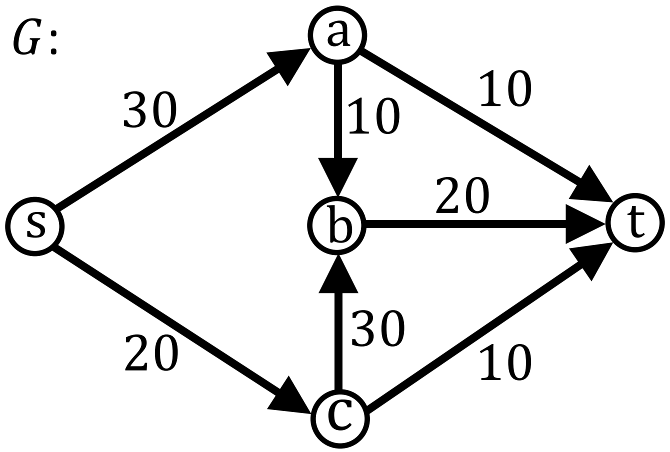

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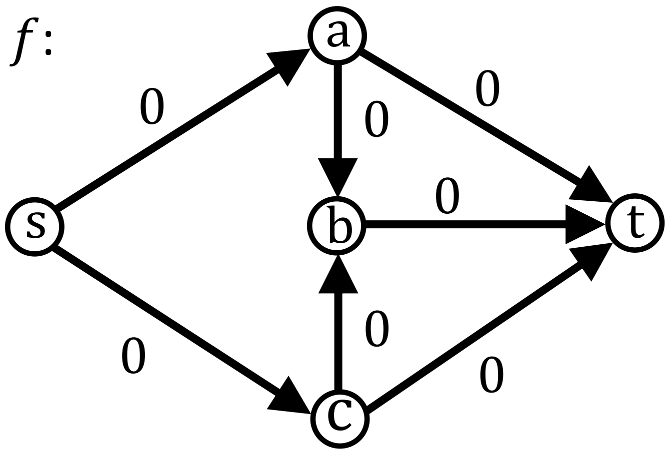
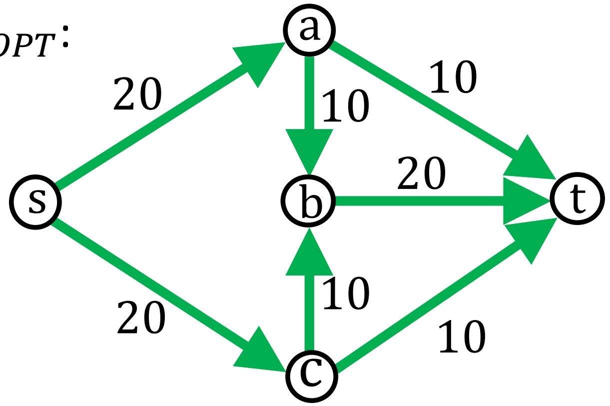


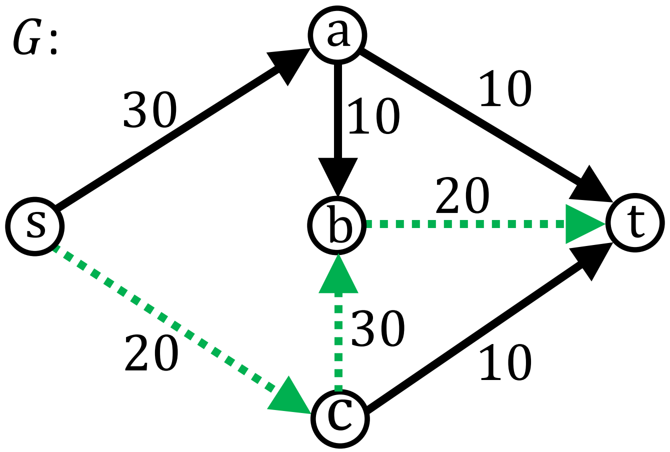
Ideas?

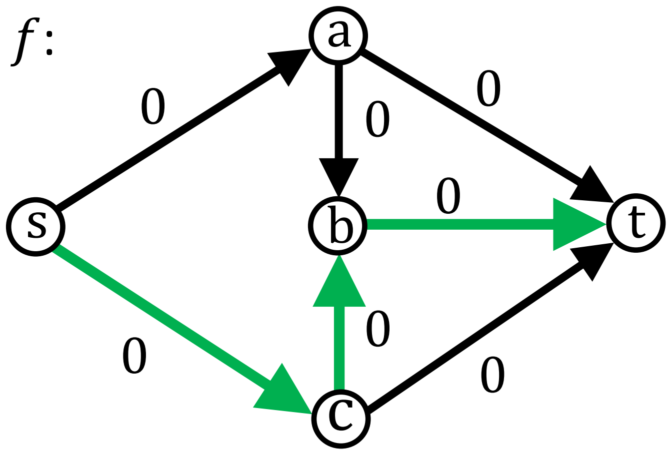
1. Select a path P.

f_{OPT}:



Max Flow Algorithm

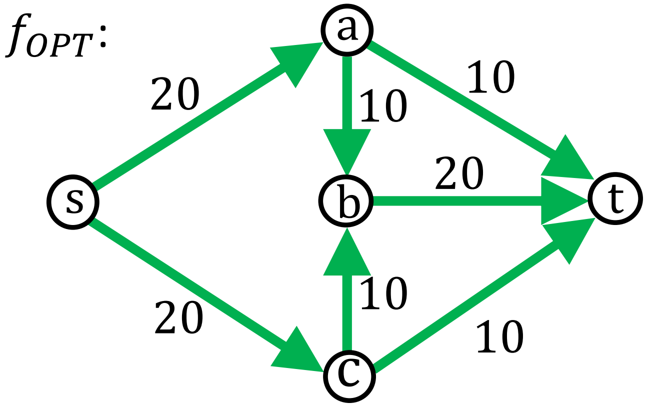




Ideas?

1. Select a path P.

How much flow should we push?



Max Flow Algorithm

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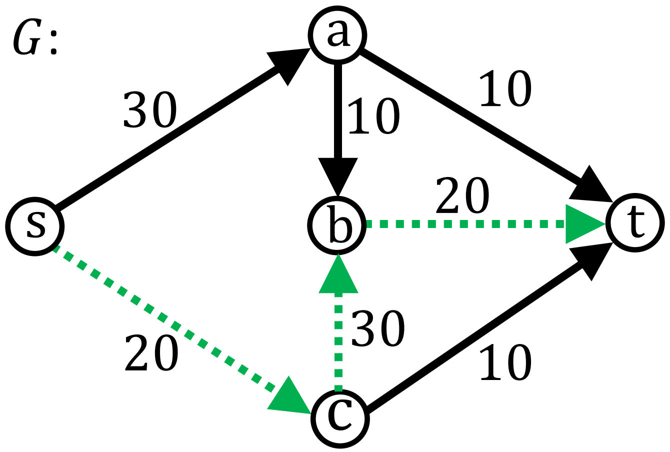

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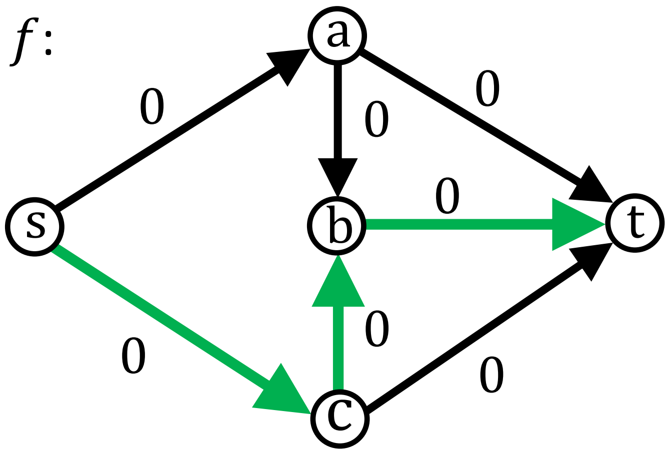


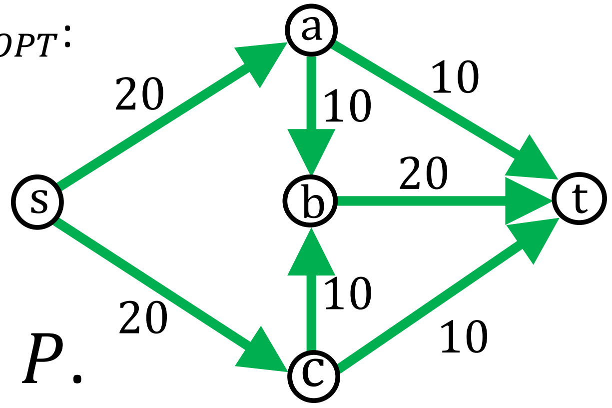
Ideas?

1. Select a path P.

How much flow should we push?

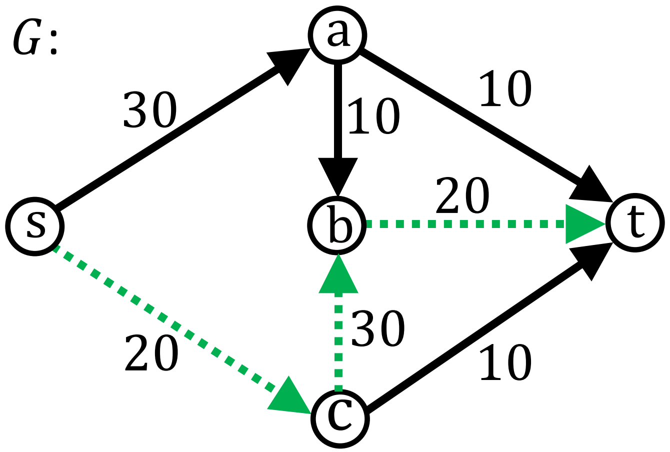
As much as possible.

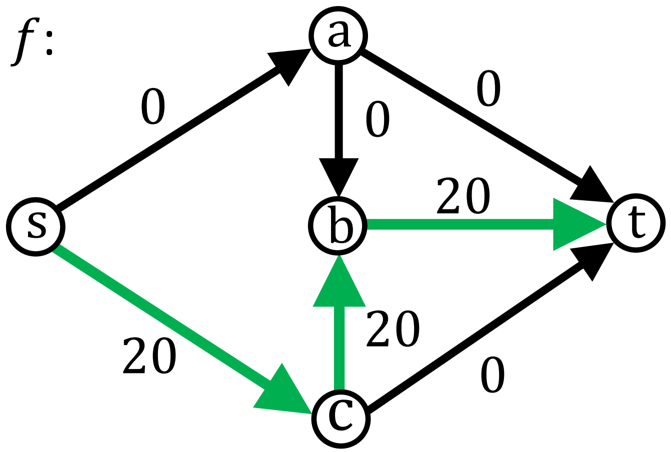
f_{OPT}:



`bottleneck(P)` = minimum capacity on any edge in path P.

Max Flow Algorithm





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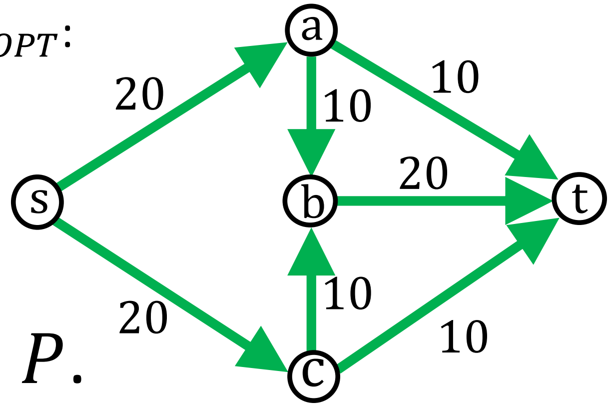

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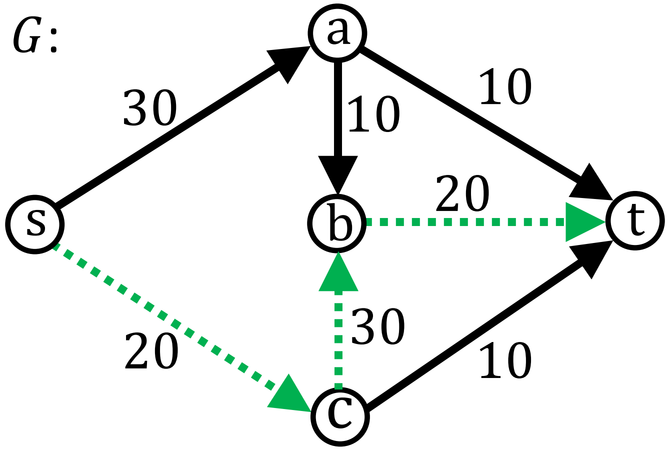

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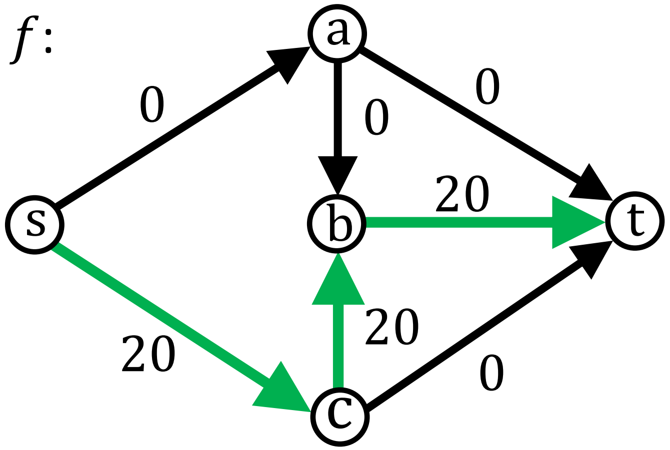

Ideas?

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2. Push bottleneck(P) flow on P.

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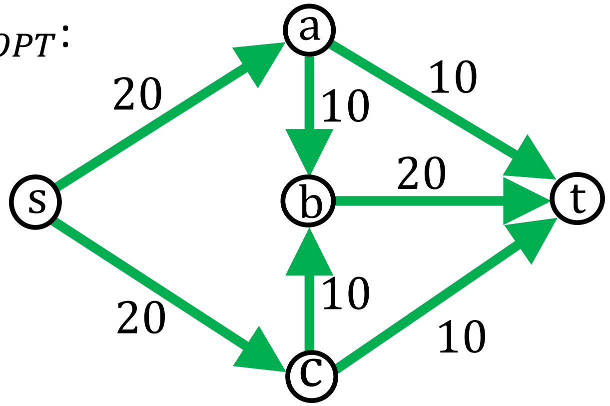

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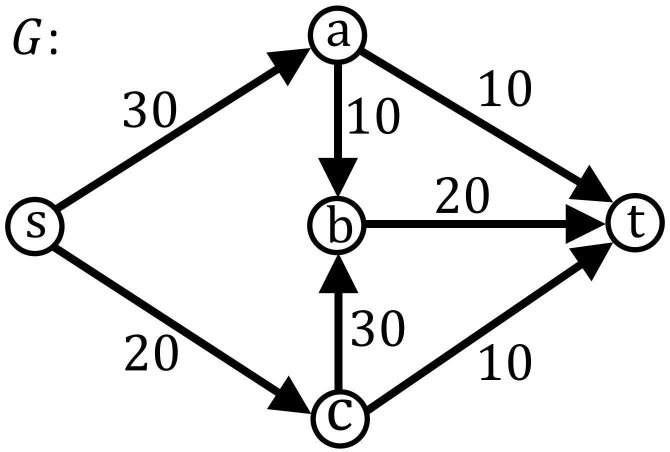

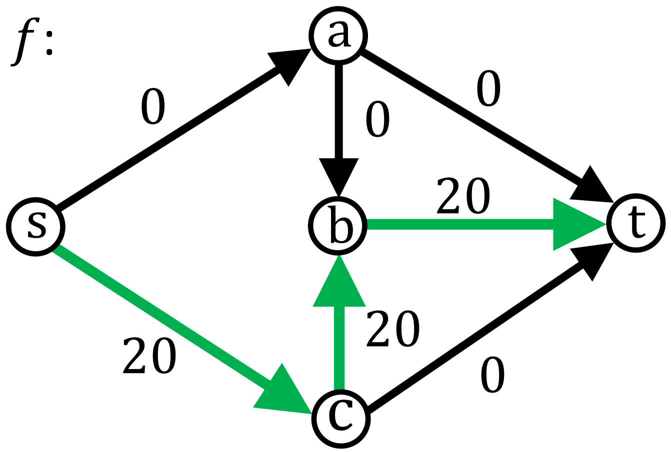

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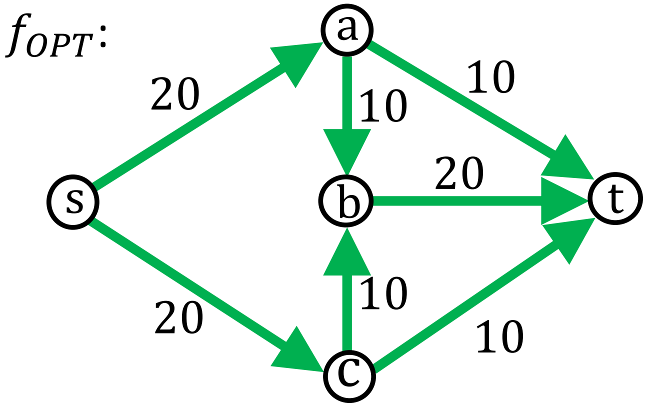
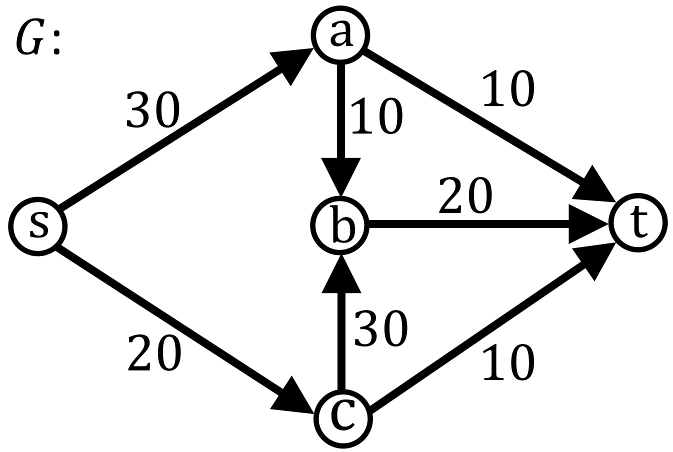
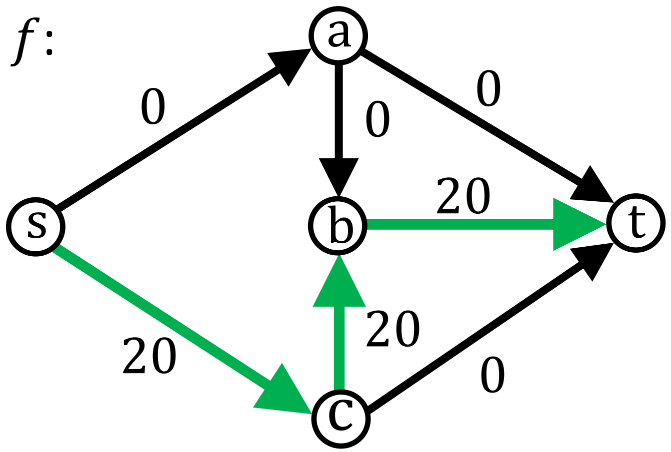
So far:
- **Guaranteed to meet conservation of flow constraints.**
- **Guaranteed to meet capacity constraints.**

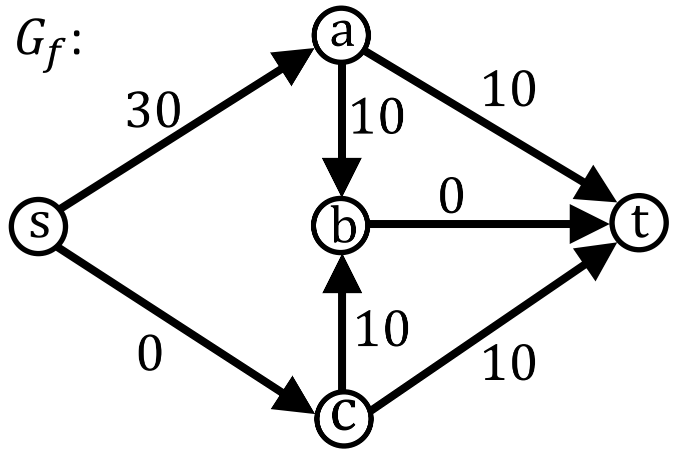


Max Flow Algorithm

Residual Capacity







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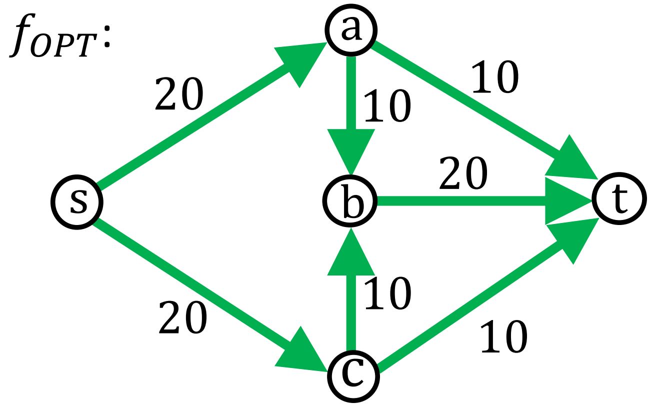

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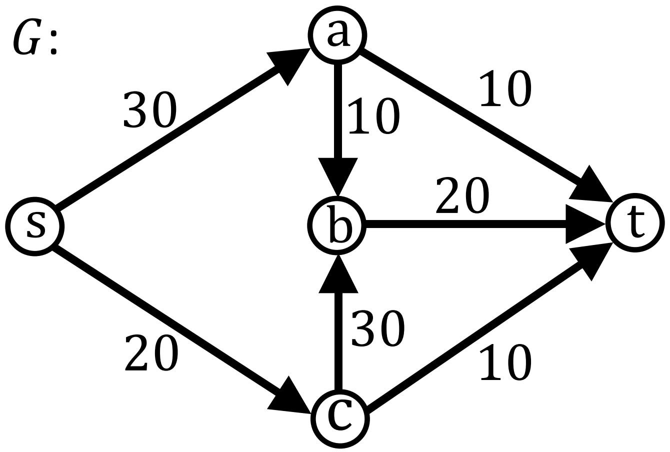

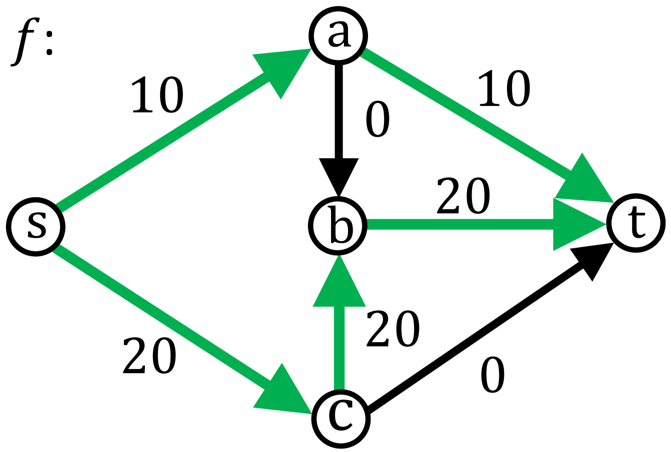

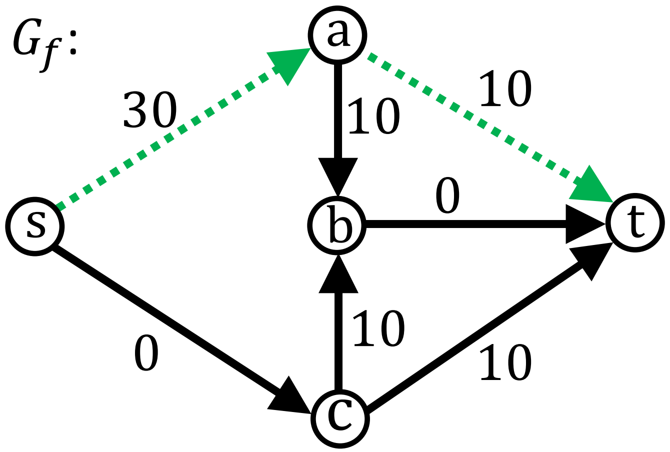

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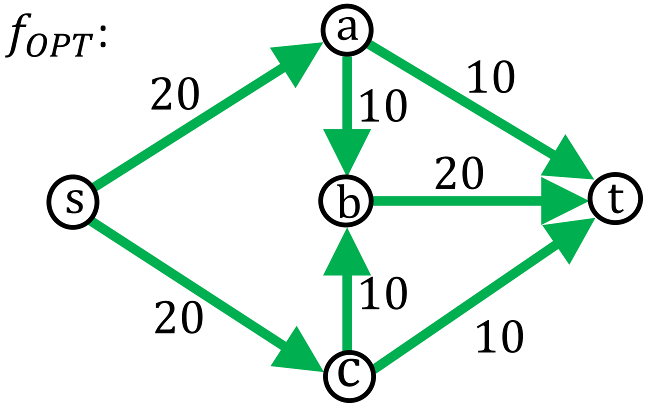
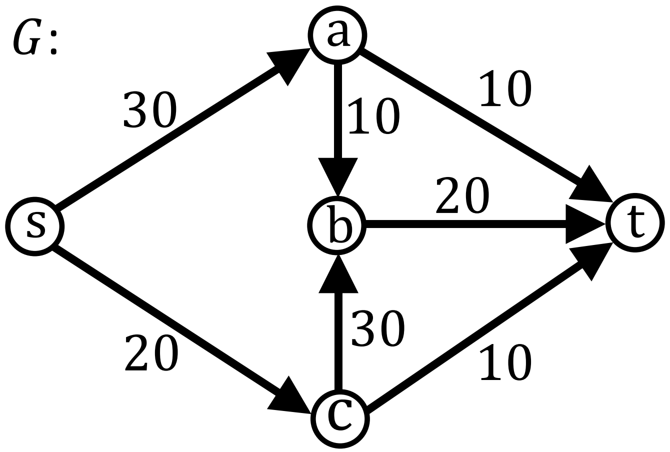
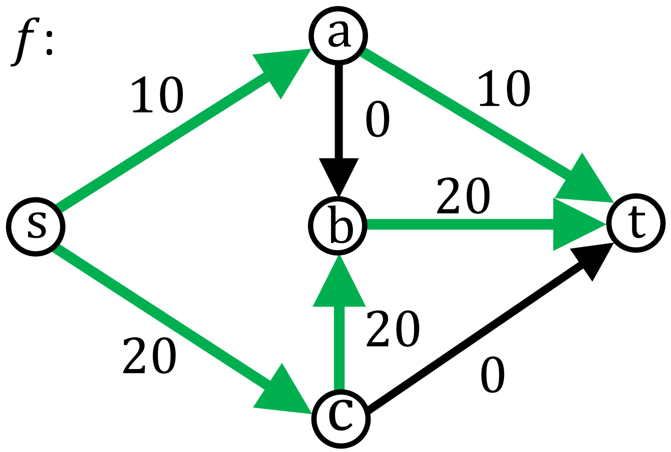
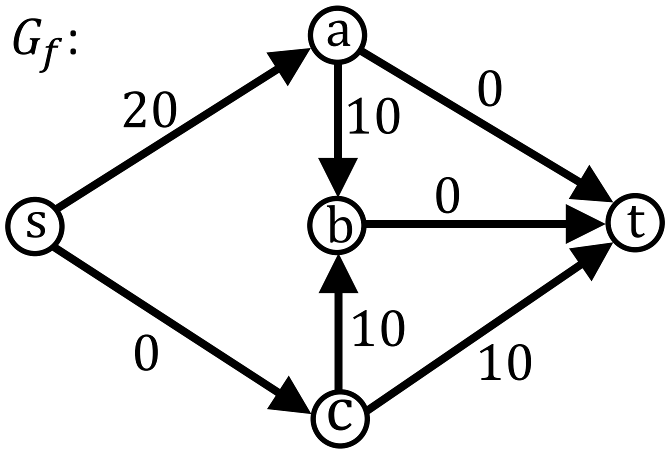


Max Flow Algorithm







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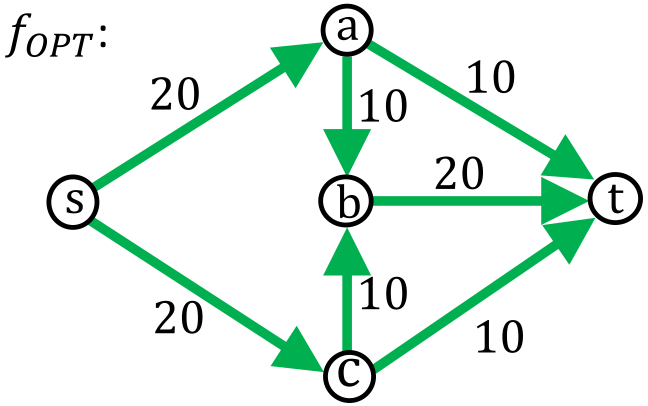
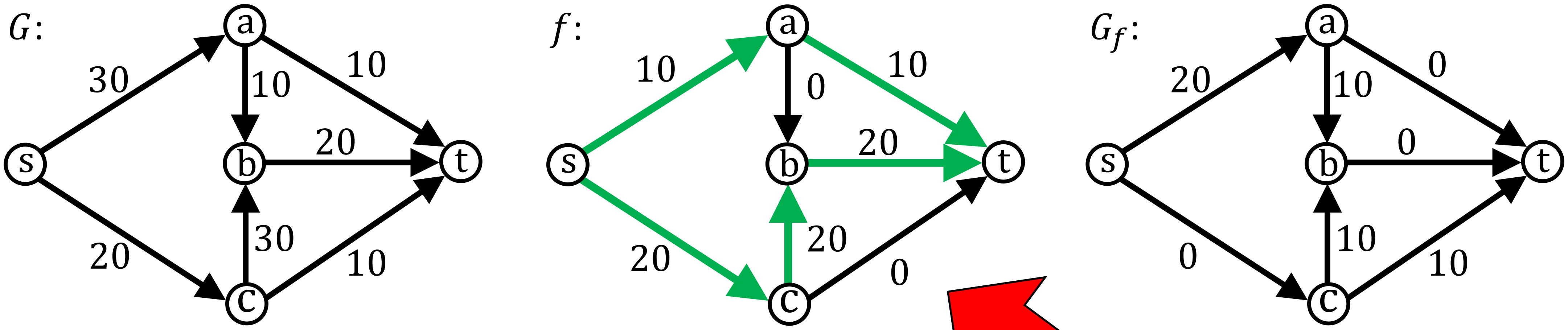
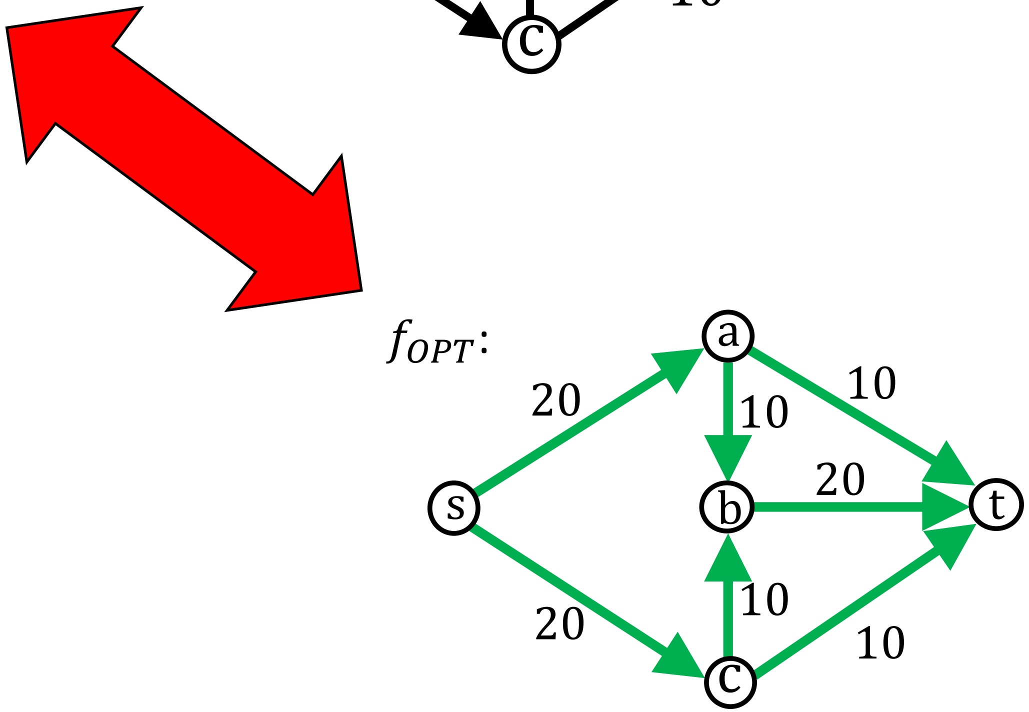


Max Flow Algorithm



We need some way to reroute flow.



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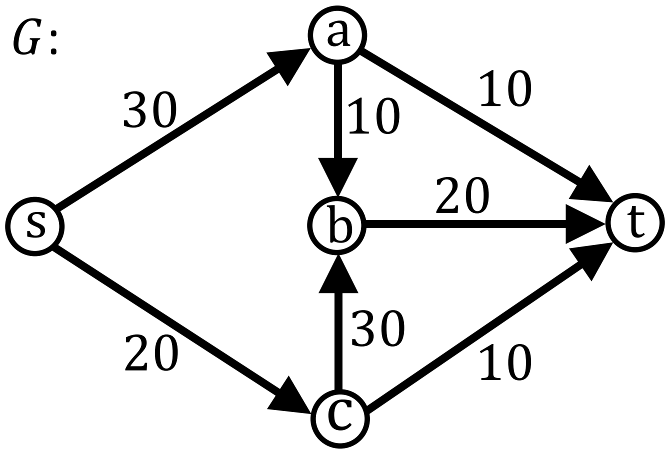

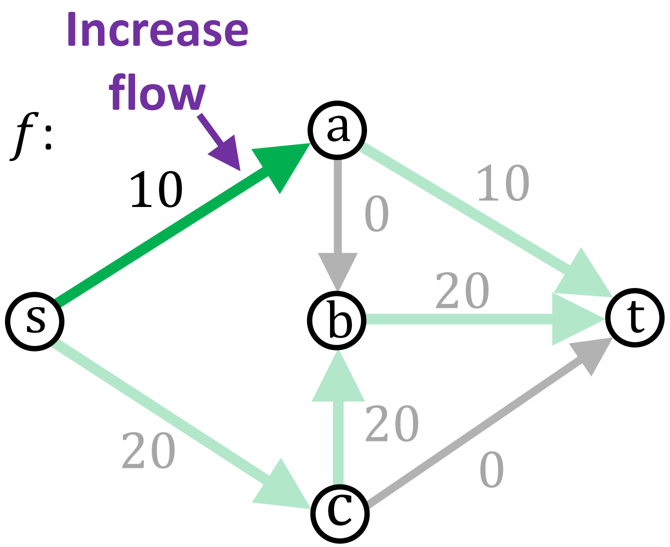

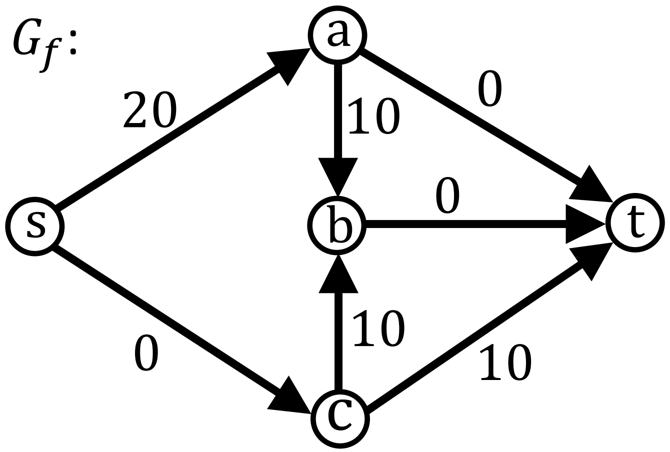

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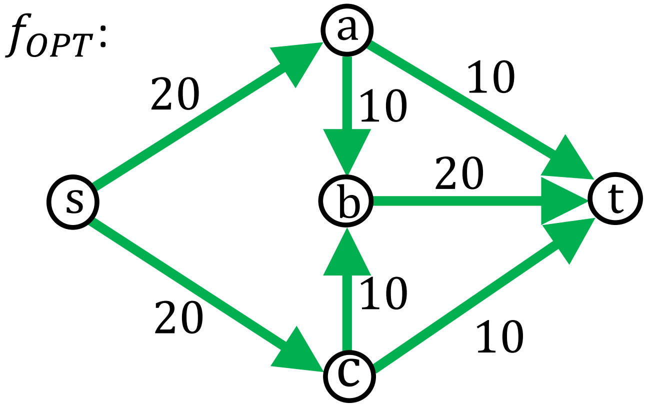

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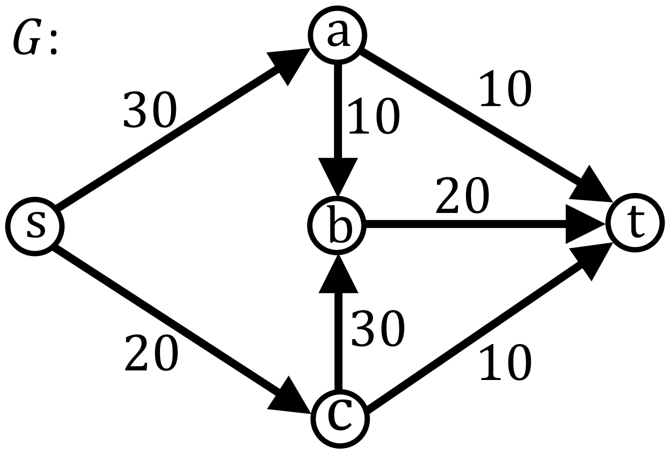

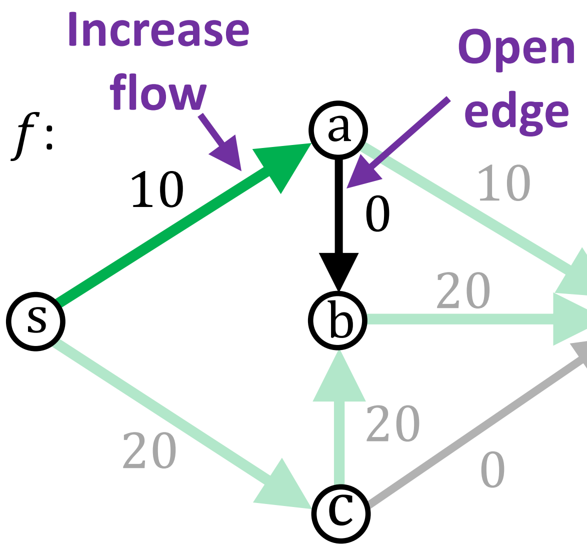

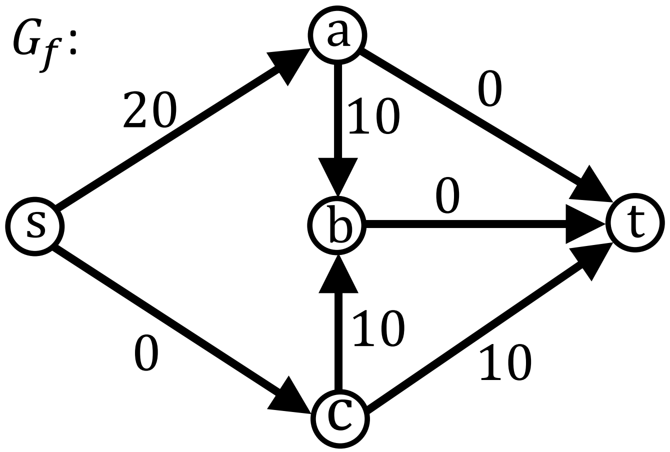

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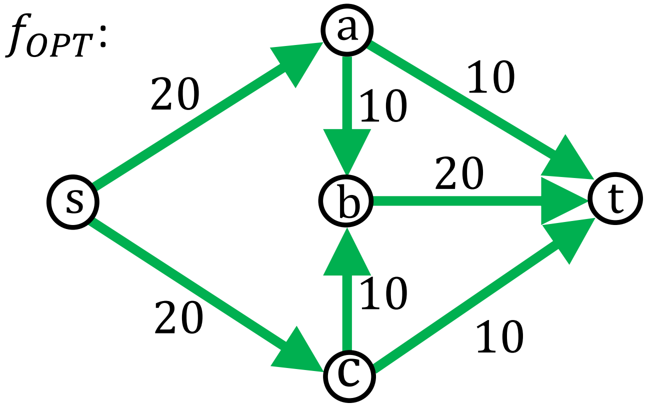
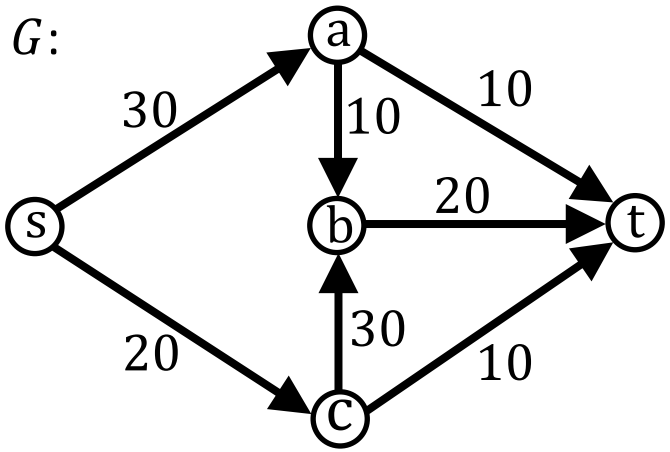


Max Flow Algorithm

G:



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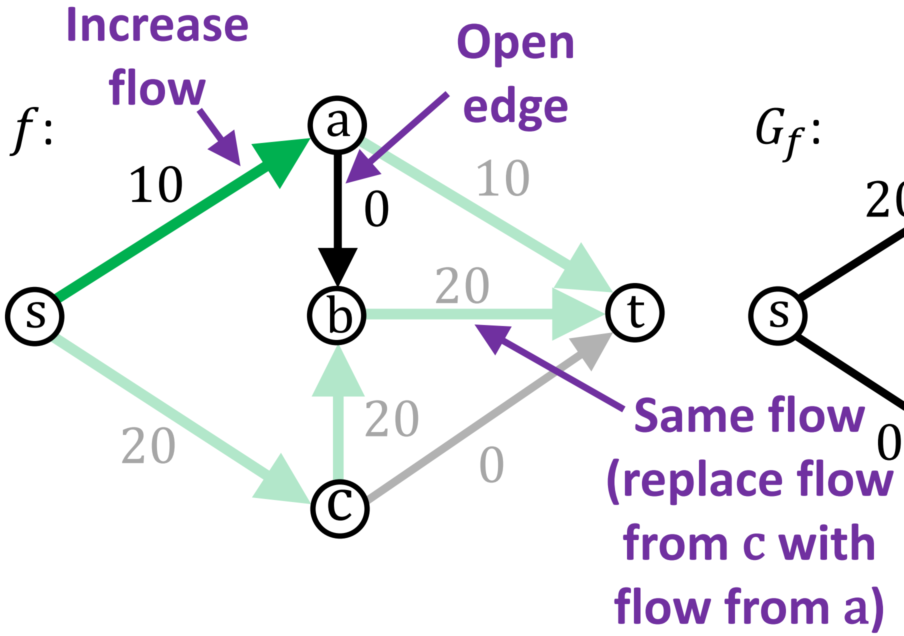
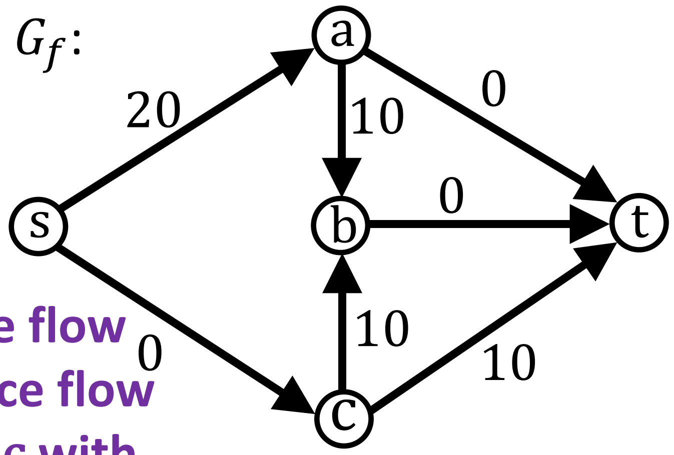


G_f:



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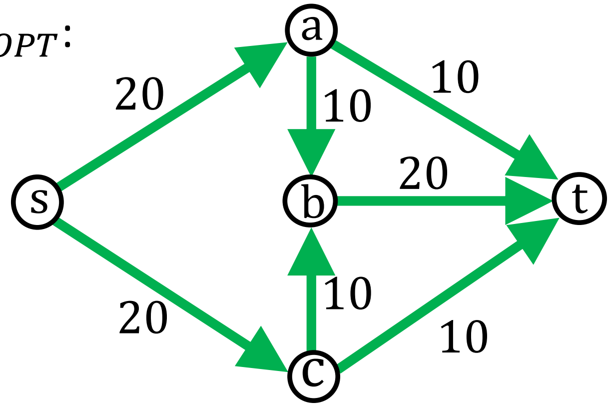

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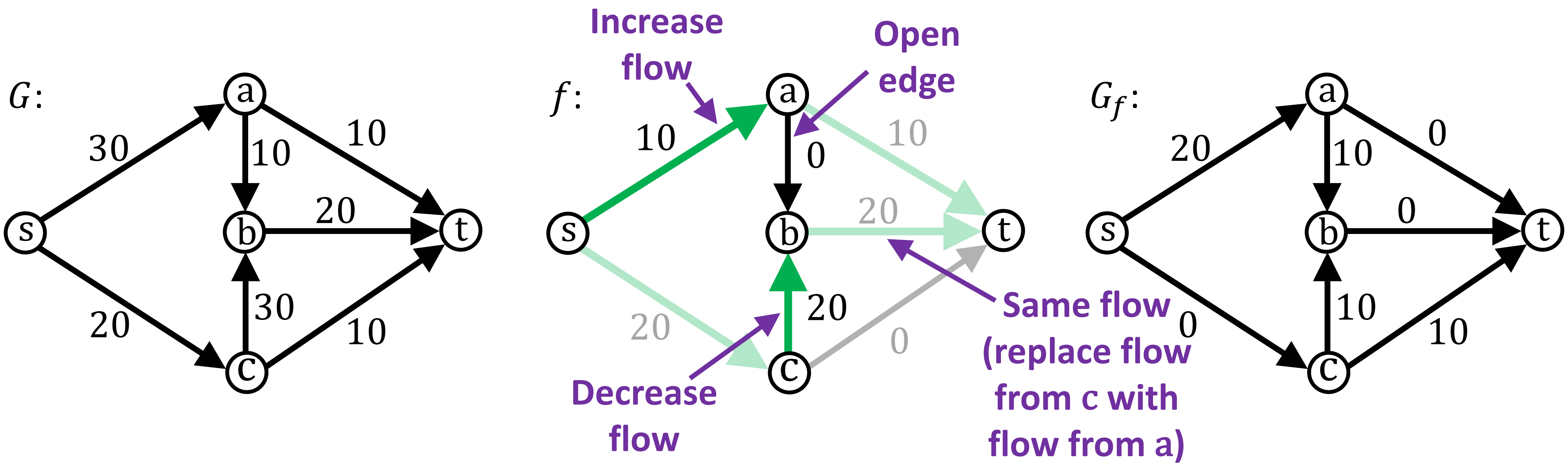

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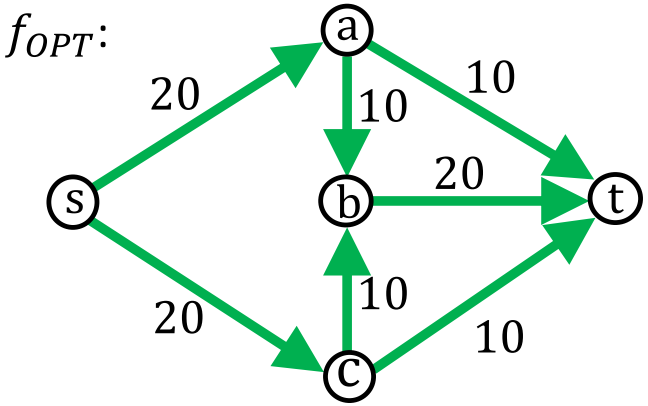

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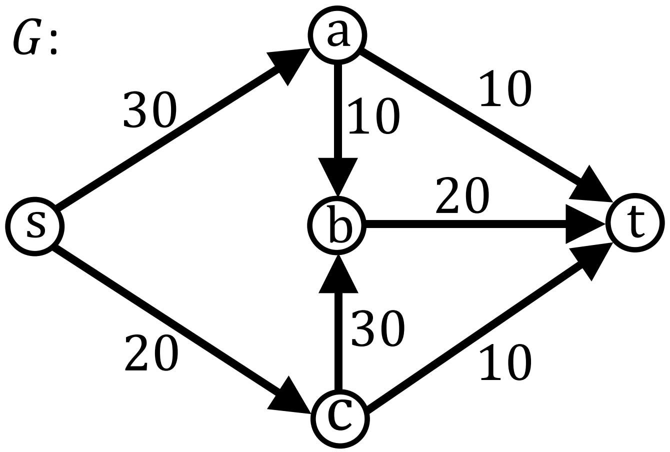
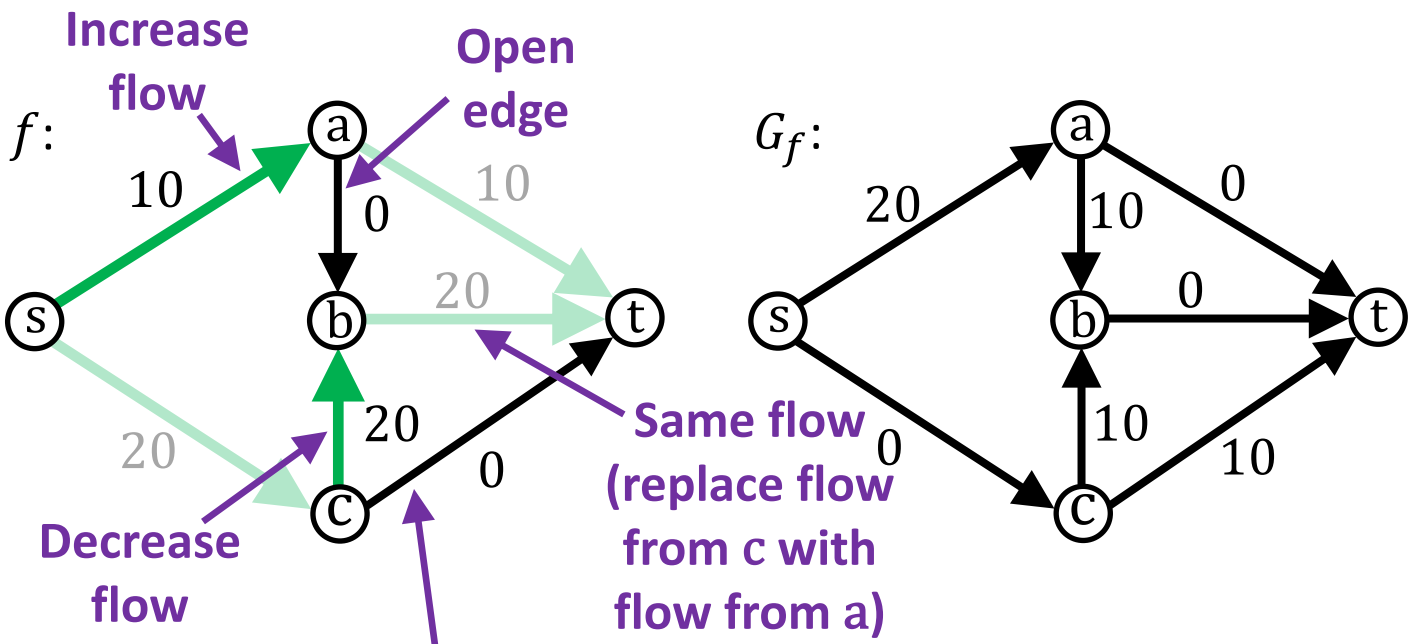
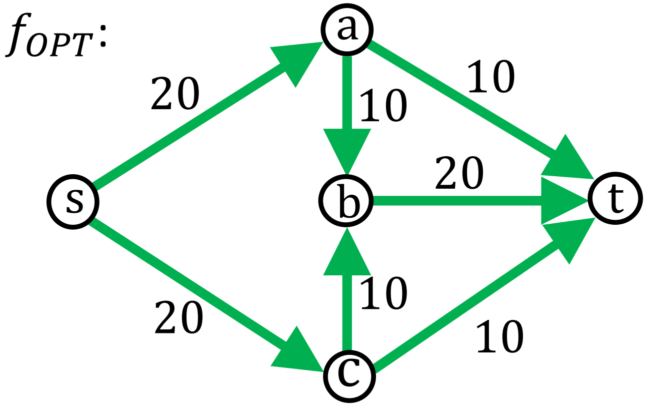




We need some way to reroute flow.

Open edge (using flow from c that was not sent to b)



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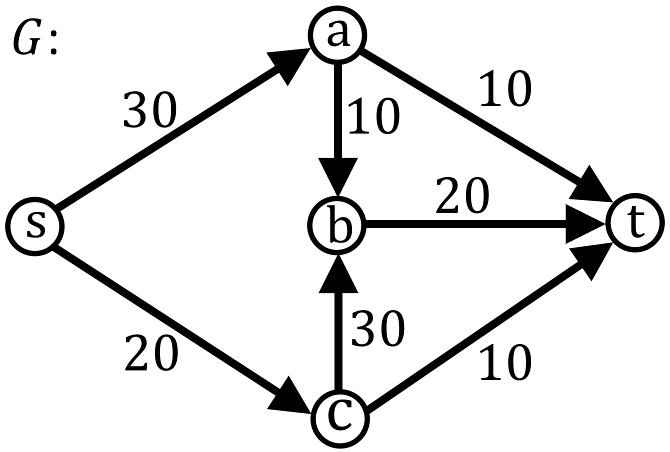

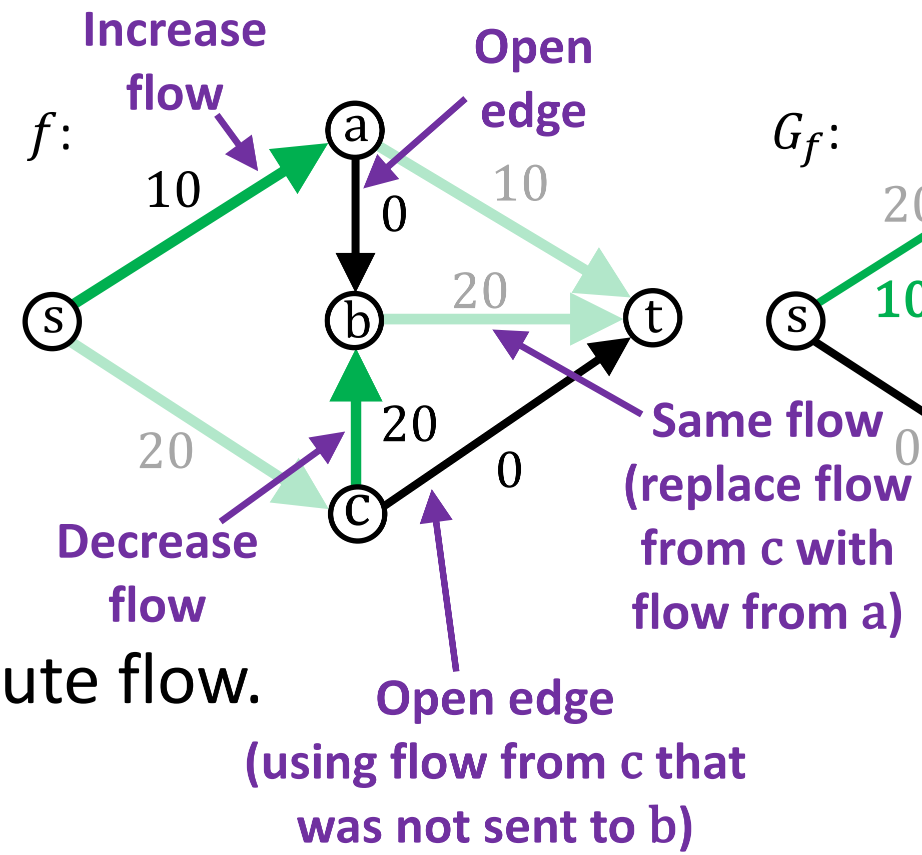

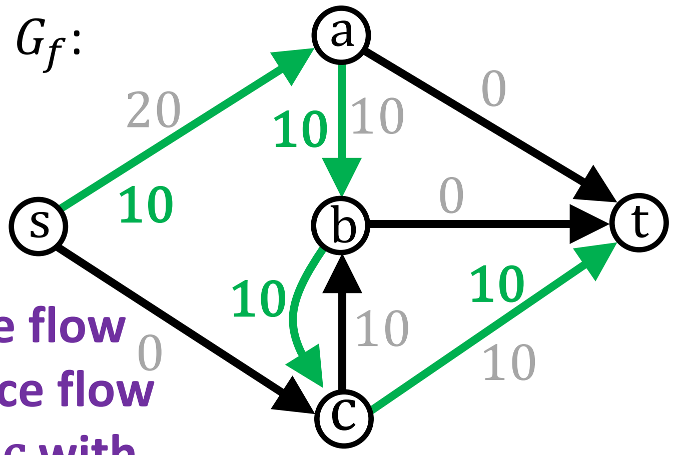

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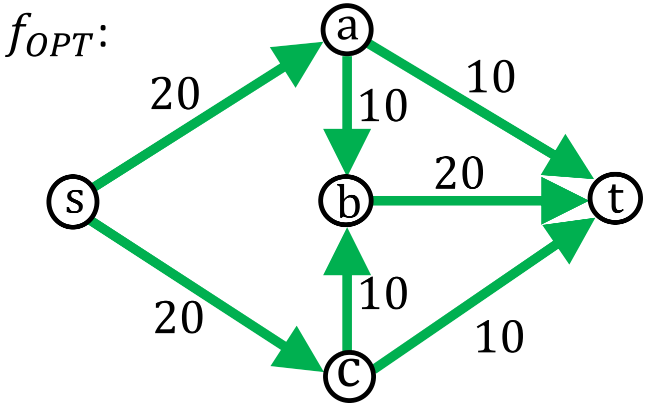

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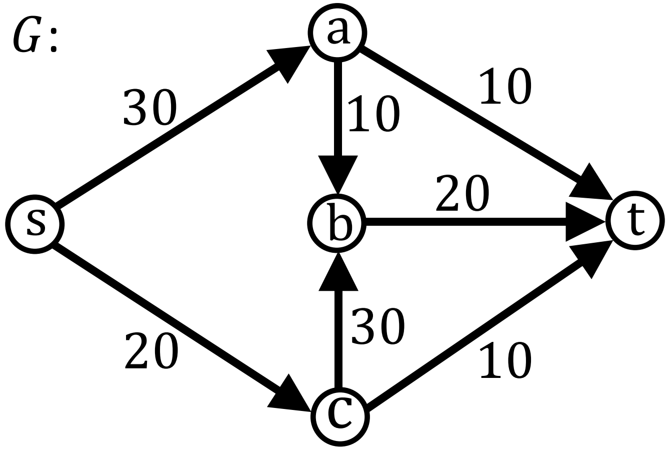

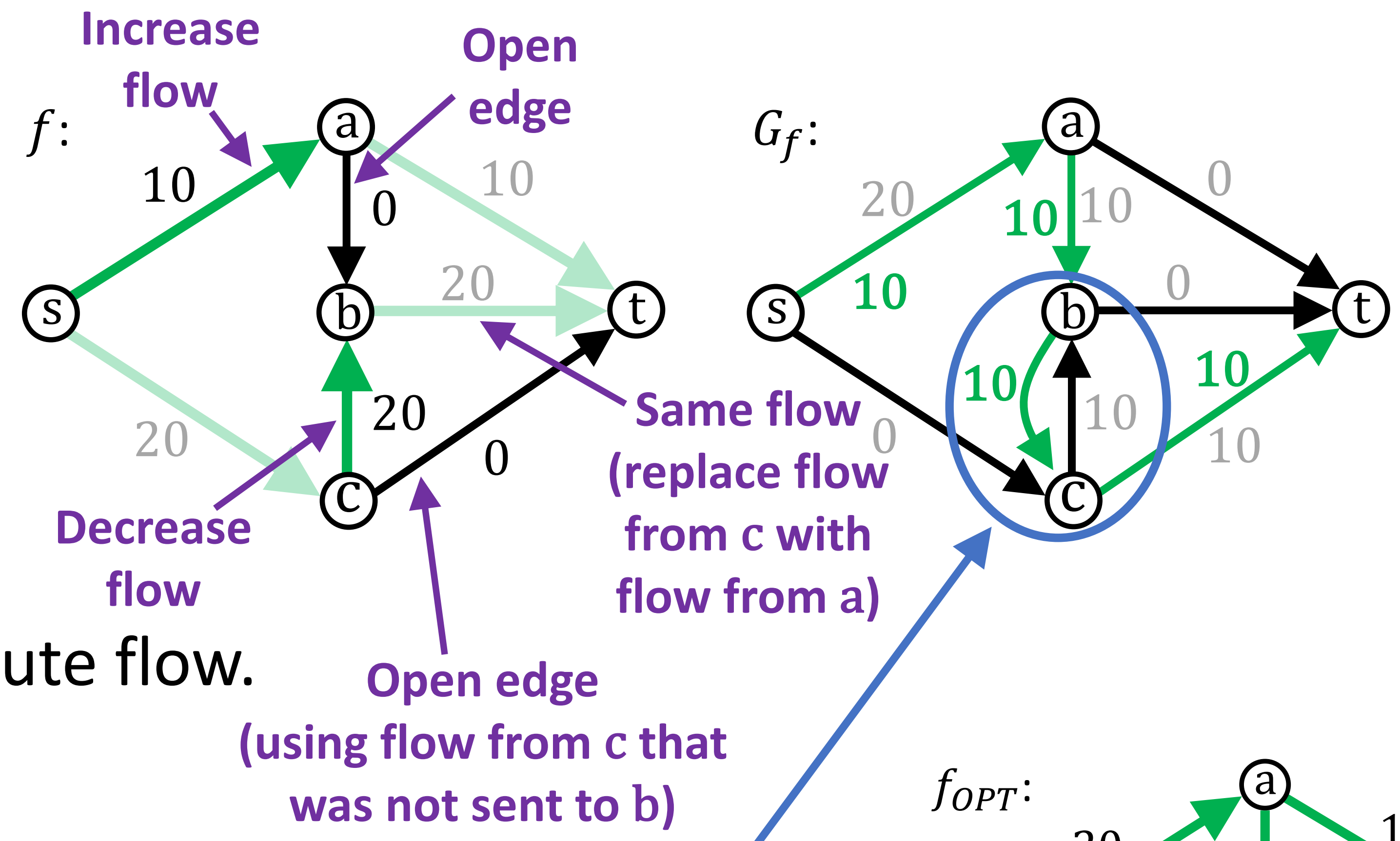

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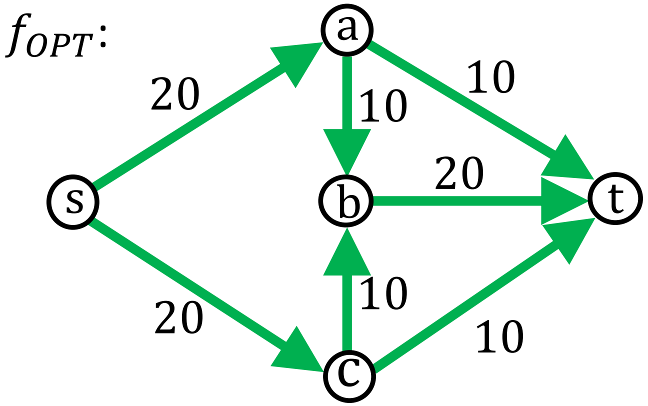
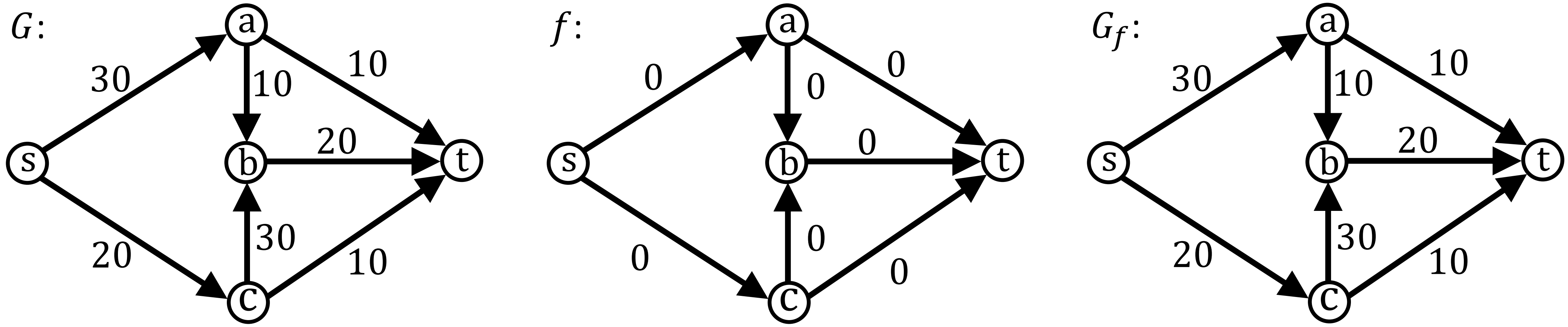


This edge does not physically exist but exists virtually as a mechanism to reroute flow that has already been pushed.

Max Flow Algorithm



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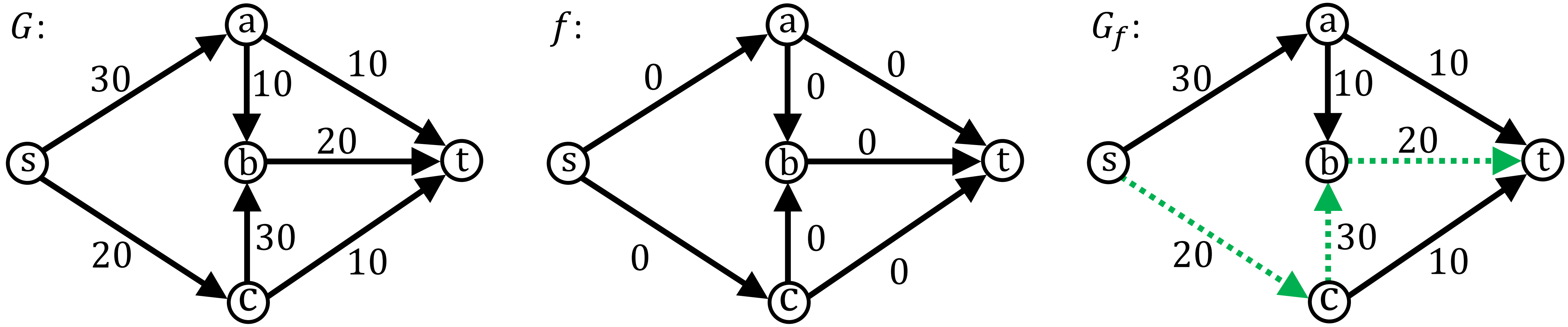

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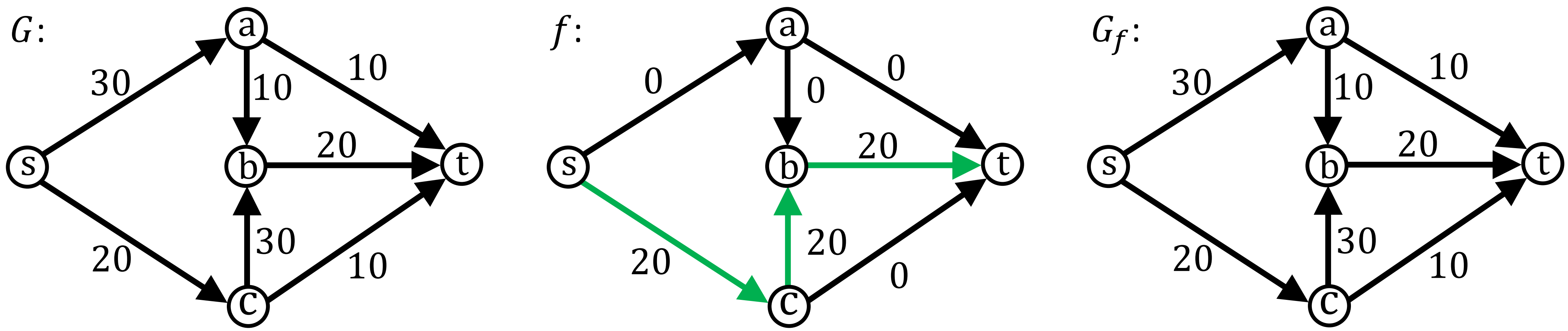

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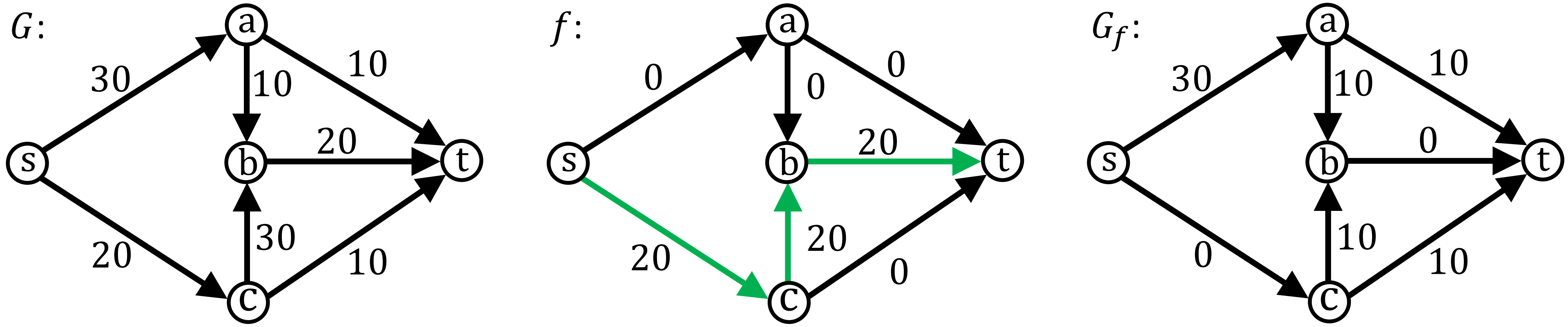

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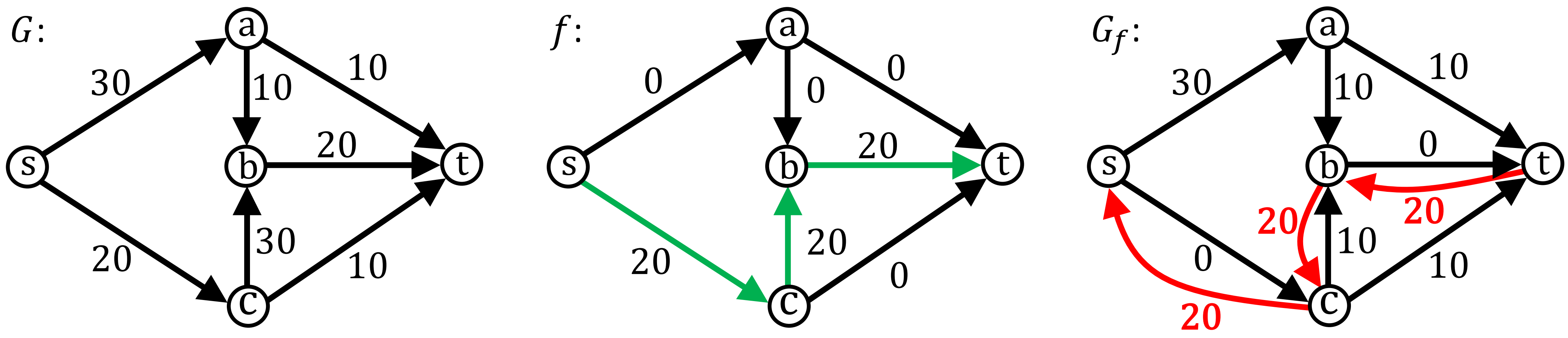

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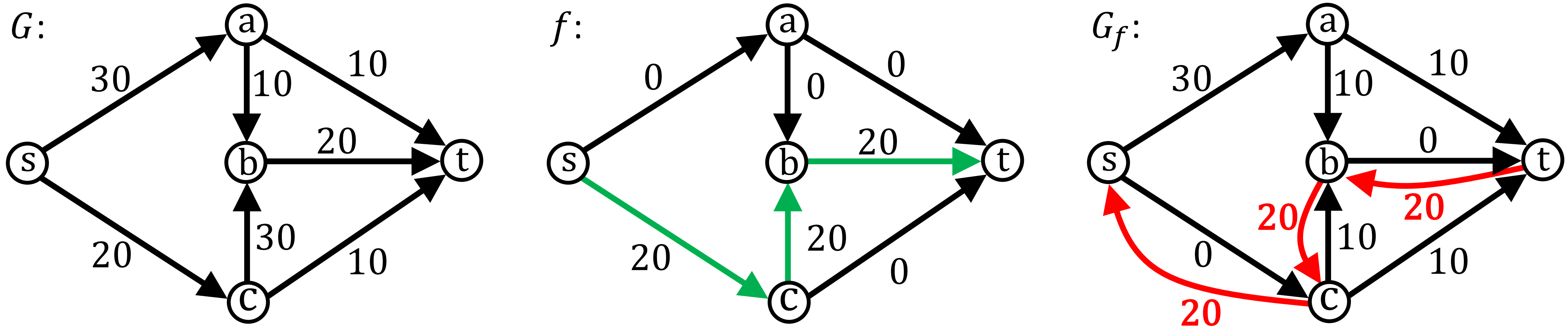

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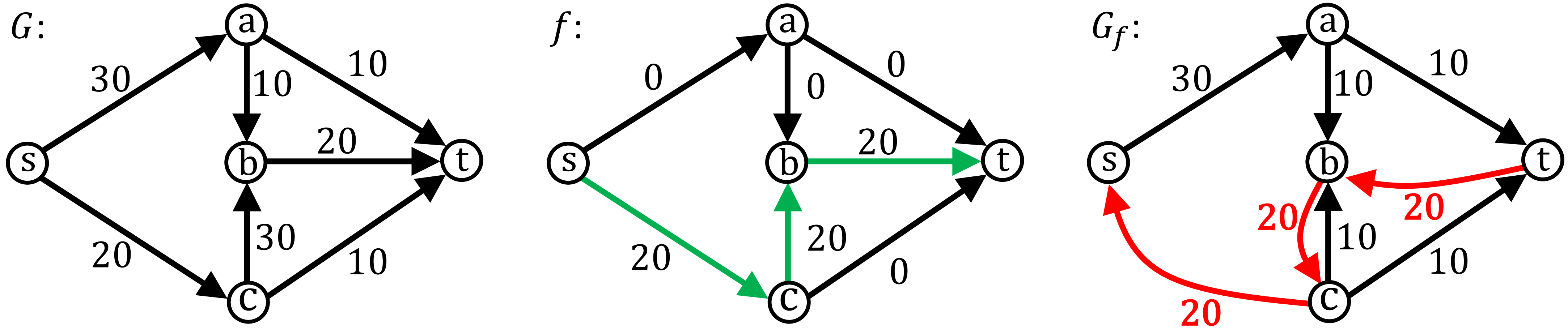
Remove edges with 0 capacity in residual graph. Can't use them anyway.

Max Flow Algorithm



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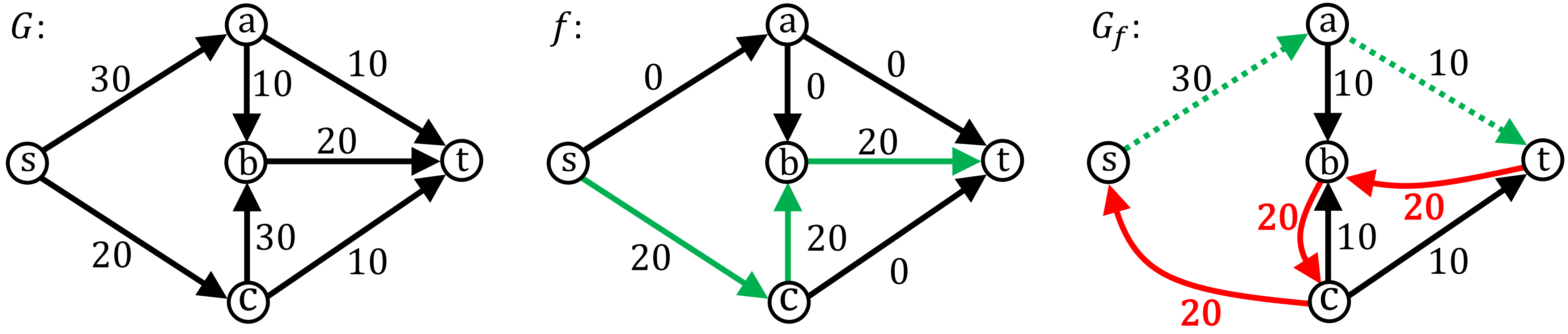

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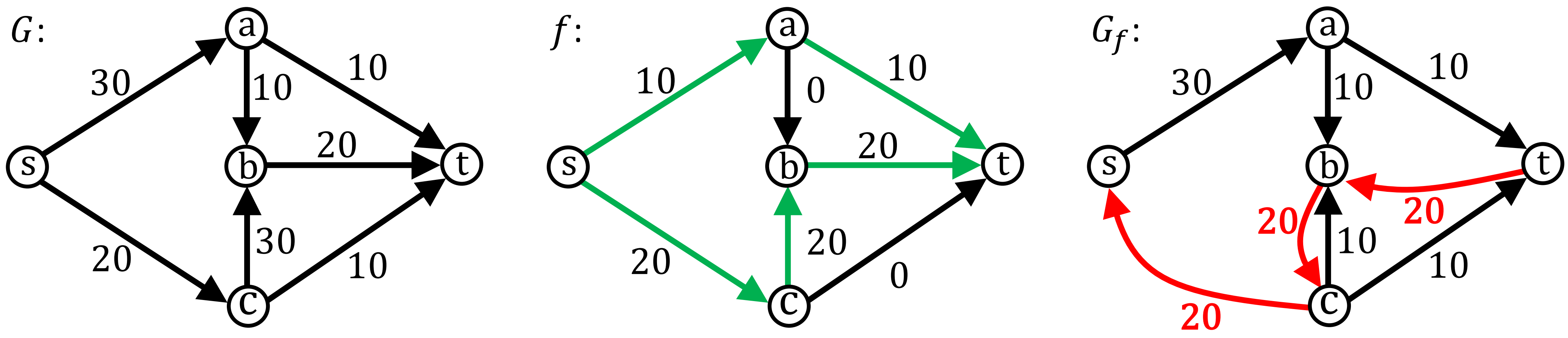

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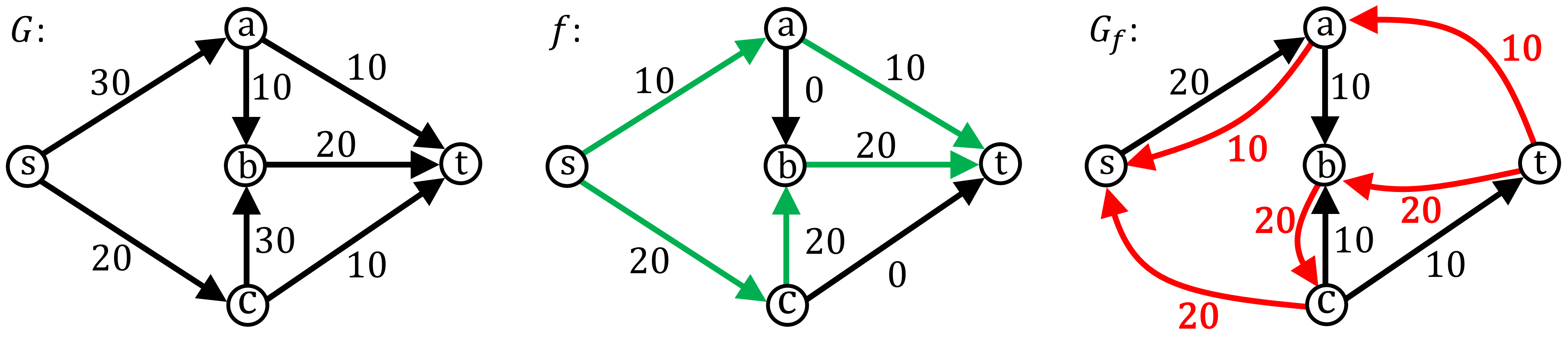

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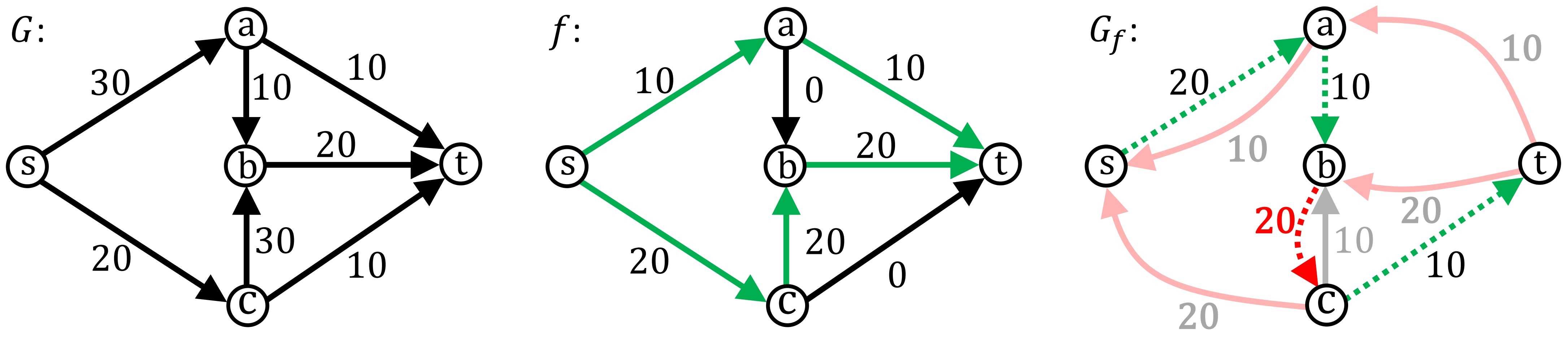

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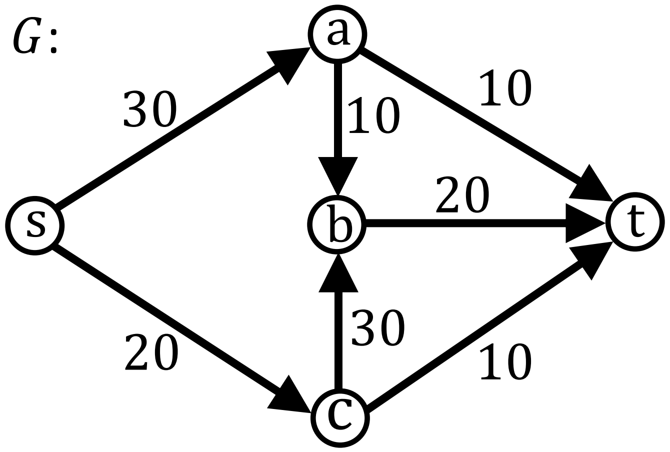

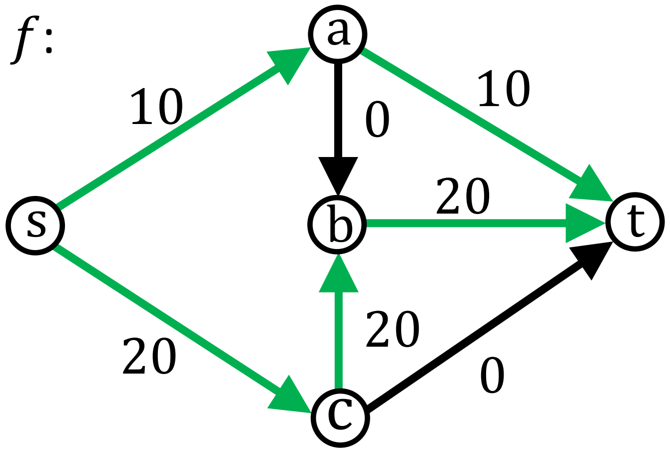

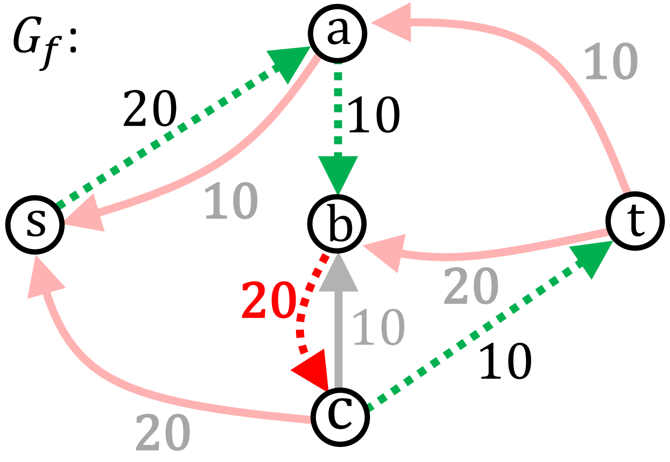

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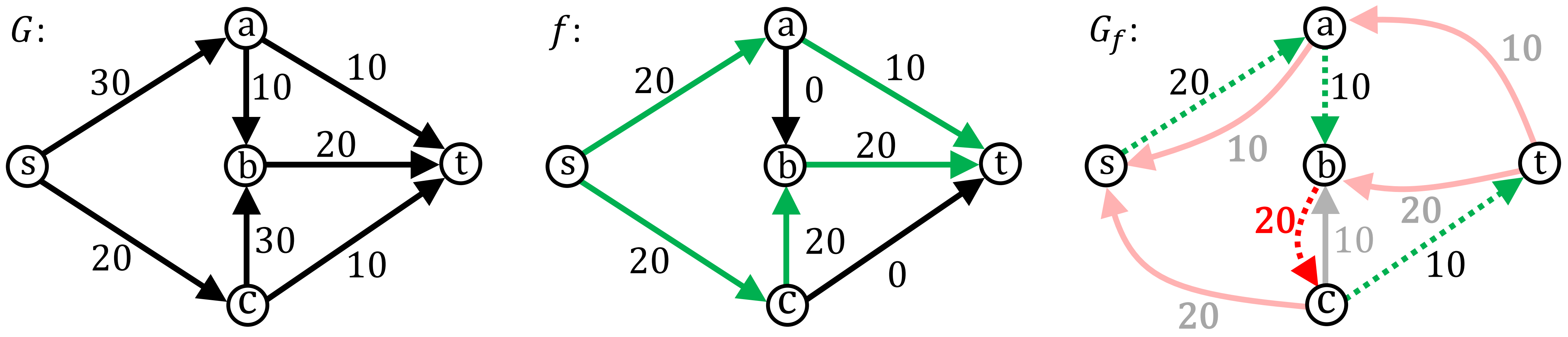
Pushing flow on a path:

Max Flow Algorithm



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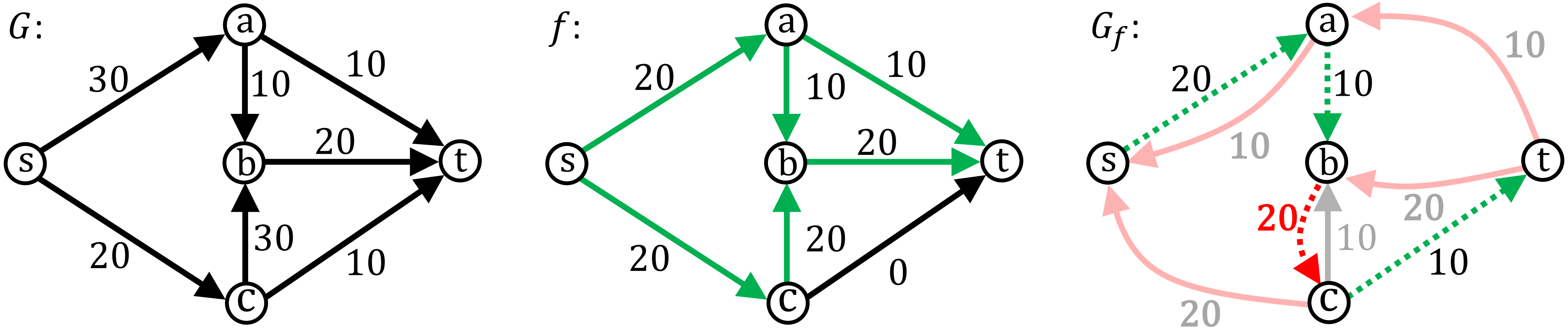
1. If edge is a normal edge, increase flow.

Max Flow Algorithm



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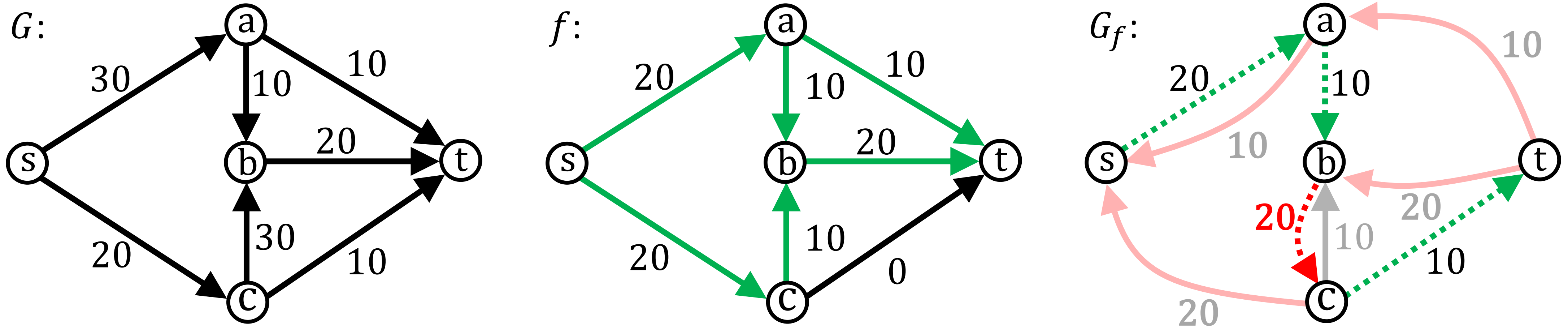

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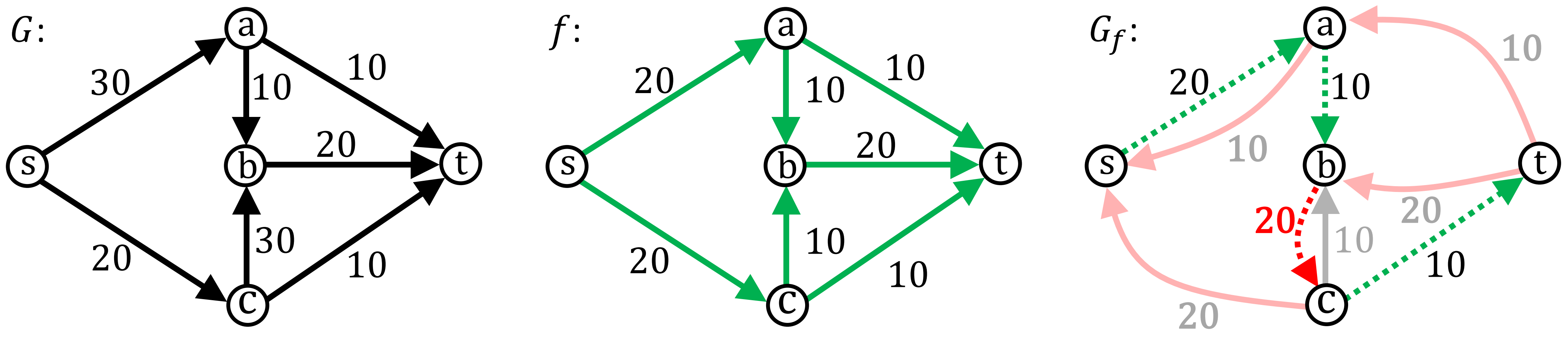

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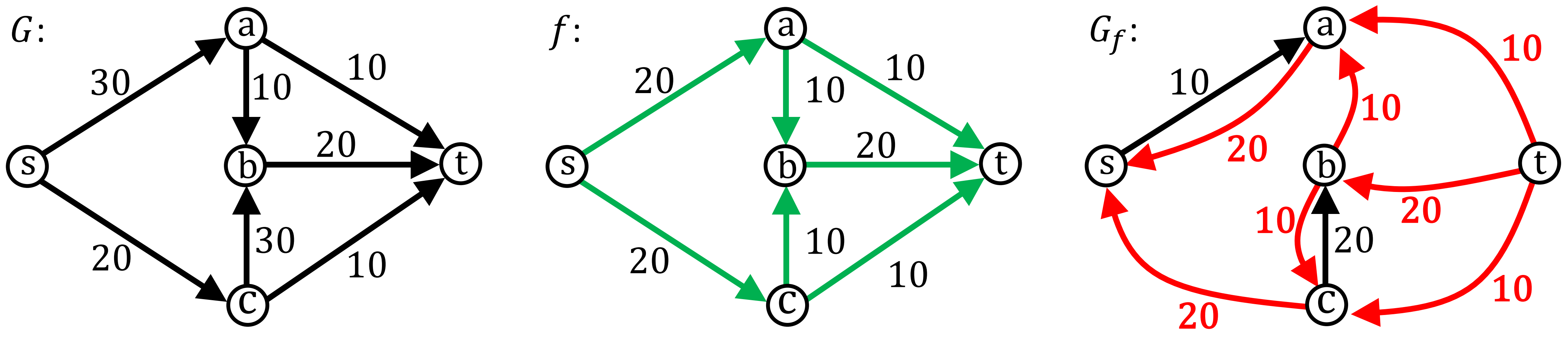

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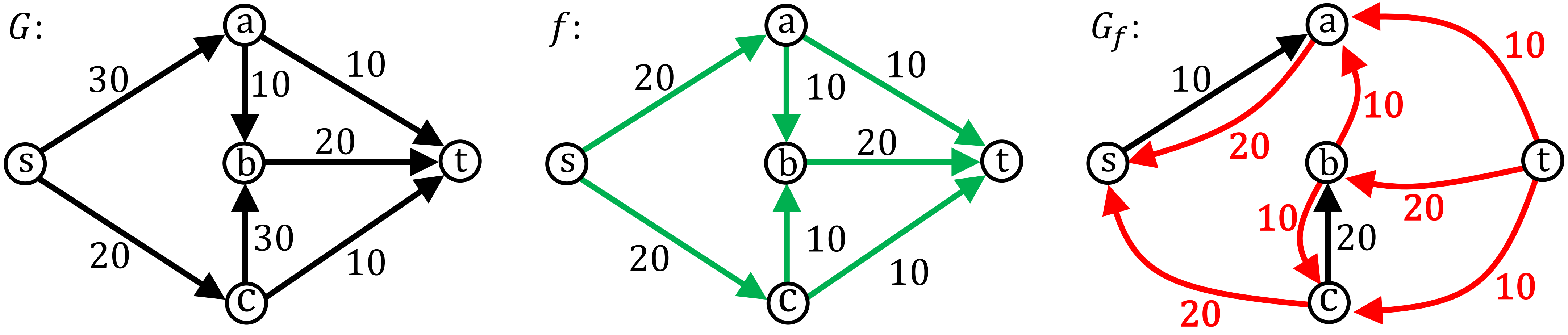

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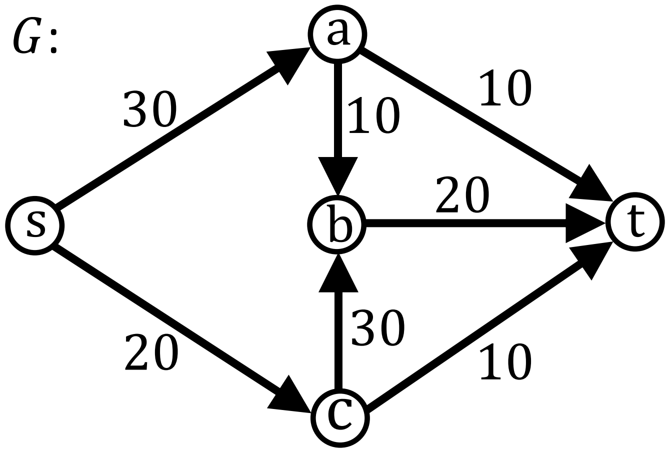

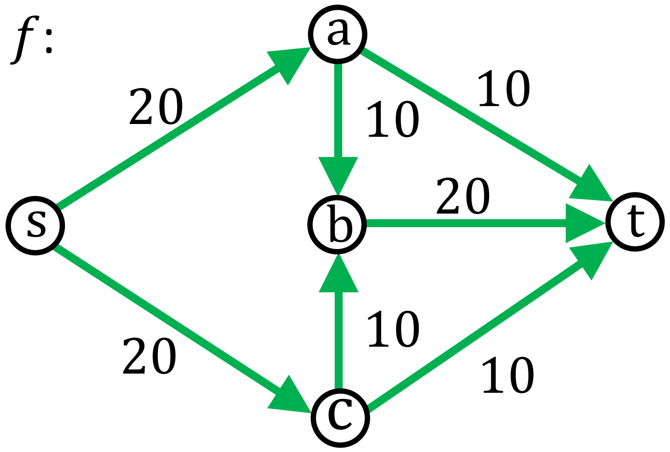

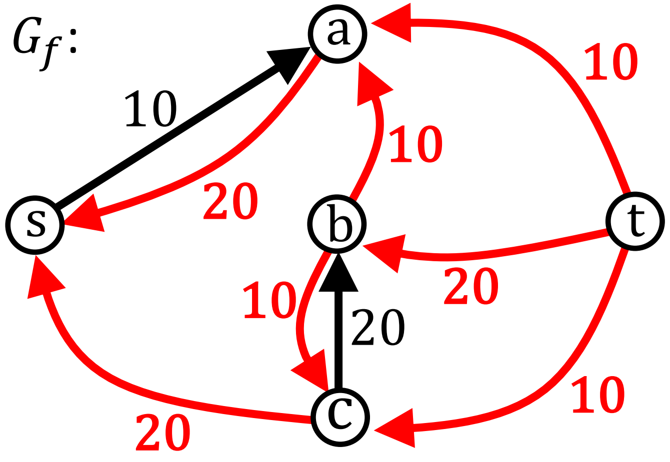

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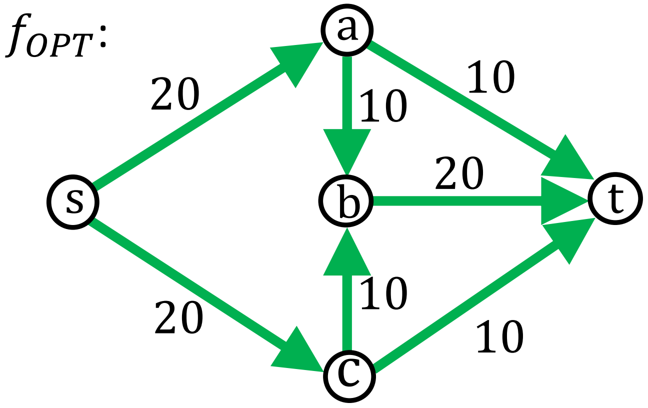
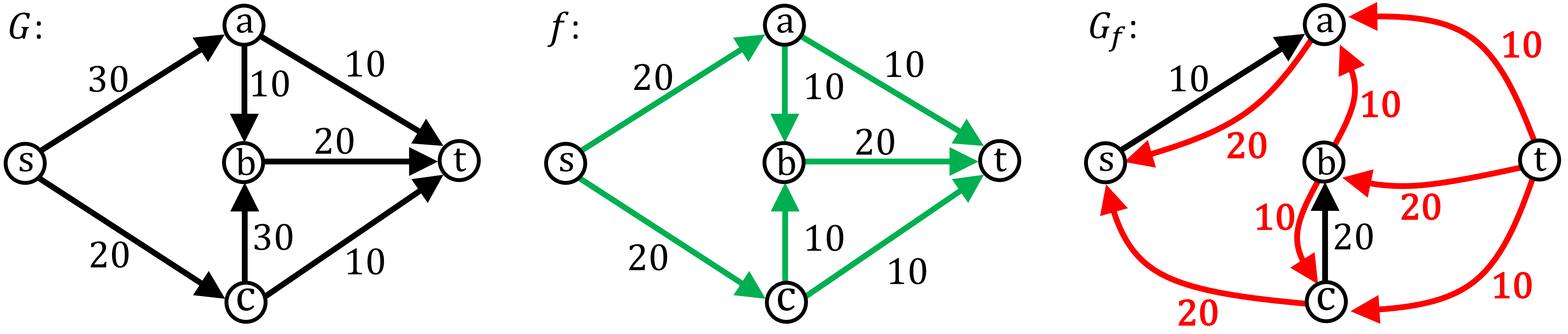


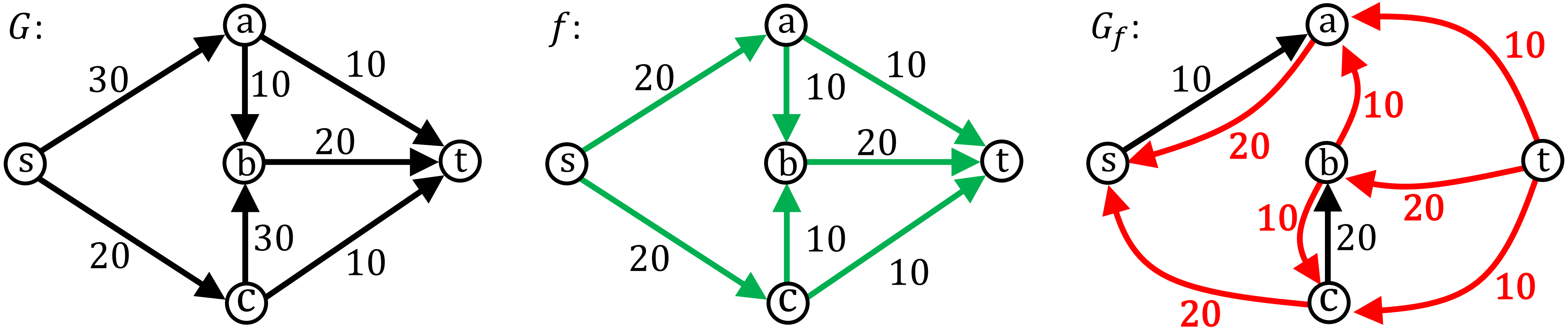
Ford-Fulkerson




```
Max-Flow(G)
   f(e) = 0 for all e in G
   while s-t path in Gf exists
      P = simple s-t path in Gf
      f'= augment(f, P)
      f = f'
      Gf = Gf'
   return f
```

```
augment(f, P)
   b = bottleneck(P,f)
   for each edge (u, v) in P
      if (u, v) is a back edge
         f((v, u)) -= b
      else
         f((u, v)) += b
   return f
```

Ford-Fulkerson

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Need to show:

1. Validity.
2. Running time.
3. Finds max flow.

Ford-Fulkerson

Claims:

1. If edge capacities are integer-valued, flows found by algorithm will be too.

2. If edge capacities are integer-valued, the algorithm will terminate.

3. If an iteration starts with a valid flow, it ends with a valid flow.

4. The first iteration starts with a valid flow.

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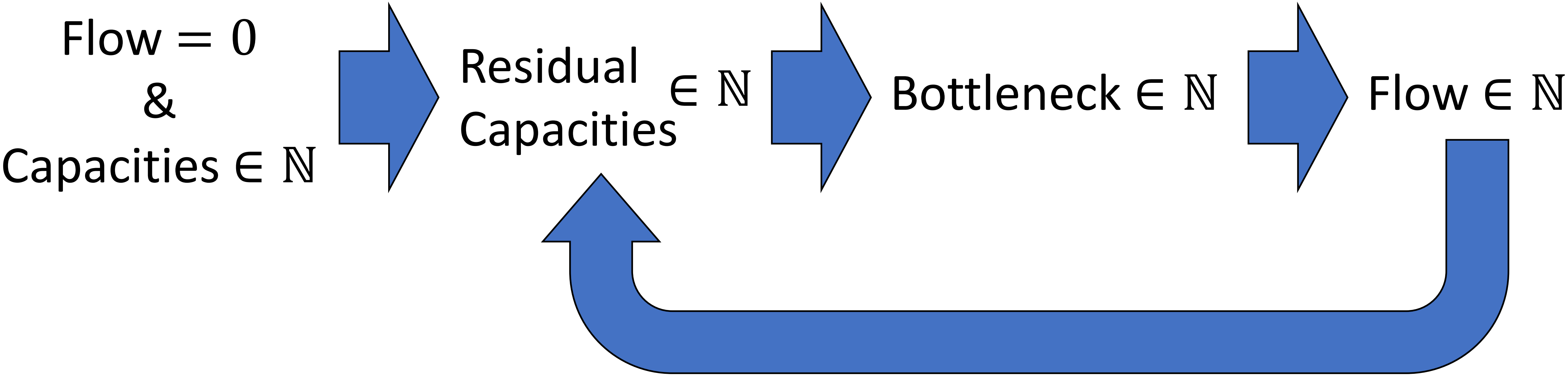

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Bottleneck $\in \mathbb{N}$

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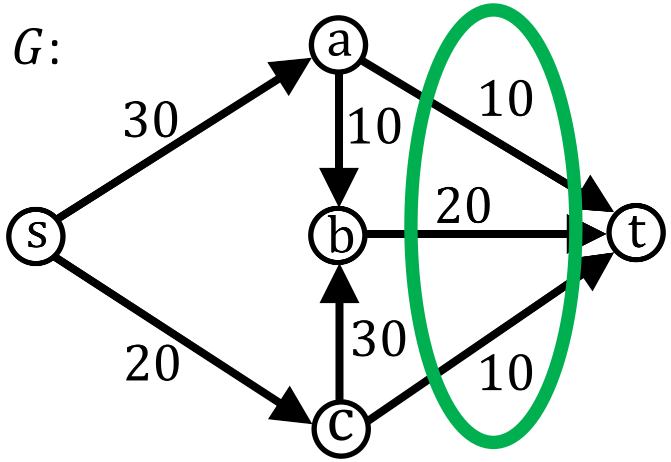

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Note: This does not hold for general edge capacities
(i.e., irrational edge capacities *can* lead to non-terminating scenarios).

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