Randomized Rounding
CSCI 532

Set Cover

Set Cover: Given a set of elements (the universe), and sets containing those elements, find the smallest number of sets so that every element of the universe is included.

Example:

$$U = \{1, 4, 7, 8, 10\}$$

$$S = \{\{1, 7, 8\}, \{1, 4, 7\}, \{7, 8\}, \{4, 8, 10\}\}$$

$\{\{1, 7, 8\}, \{4, 8, 10\}\}$ ✓

$\{\{1, 4, 7\}, \{7, 8\}\}$ ✗

Set Cover

```
while element of universe not included
   select Si with largest number of
   excluded elements.
```

Suppose the universe contains n elements.
Before the t^{th} iteration, some remaining set has at least $\frac{n_{t-1}}{OPT}$ uncovered elements and the number of elements remaining after the t^{th} iteration is:

$$n_t \leq n_{t-1} - \frac{n_{t-1}}{OPT} = n_{t-1}\left(1 - \frac{1}{OPT}\right) \leq n\left(1 - \frac{1}{OPT}\right)^t$$

Trust that: $1 - x < e^{-x}$ for all $x \neq 0$

$$n_t \leq n\left(1 - \frac{1}{OPT}\right)^t < n\left(e^{-\frac{1}{OPT}}\right)^t = ne^{-\frac{t}{OPT}}$$

$ne^{-\frac{t}{OPT}} \leq 1 \Rightarrow t \geq OPT \ln n$

So, when $t = OPT \ln n$, $n_t < 1$ (i.e., no elements remain). Thus, the universe is covered after at most $t = OPT \ln n$ iterations.

$$\Longrightarrow \boldsymbol{ALG \leq \ln n \, OPT}$$

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ILP:

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ILP:

Objective:	$\min \sum_s x_s$
Subject to:	$\sum_{s:\, u \in s} x_s \geq 1$, for each $u \in U$
	$x_s \in \{0,1\}$, for each set s

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$$U = \{1, 4, 7, 8, 10\}$$

$$S = \begin{Bmatrix} \{1,7,8\}, \{1,4,7\}, \\ \{7,8\}, \{4,8,10\} \end{Bmatrix}$$

Objective: $\min x_1 + x_2 + x_3 + x_4$

Subject to:
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$x_4 \geq 1$
$x_1, x_2, x_3, x_4 \in \{0,1\}$

Vertex Cover ILP

Objective:	$\min \sum_i x_i$
Subject to:	$x_i + x_j \geq 1$, for each edge $e = (i,j)$
	$0 \leq x_i \leq 1$, for each vertex i

+ If $x_i \geq \frac{1}{2}$, add vertex i to our subset S.

Can we use Vertex Cover's rounding scheme?

What is the relationship between ALG and OPT?

$\sum x_{LP} \geq \frac{1}{2}$ ALG and $\sum x_{LP} \leq$ OPT

ALG ≤ 2 OPT

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$x_1, x_2, x_3, x_4 \in [0,1]$

$U = \{1, 2, 3, 4\}$

$S = \left\{ \begin{matrix} \{1,2,3\}, \{1,2,4\}, \\ \{1,3,4\}, \{2,3,4\} \end{matrix} \right\}$

Yes, in this case $x_s = \frac{1}{3}, \forall s \Rightarrow$ No sets are selected (invalid solution).

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$$\begin{aligned} \Pr[u \text{ not covered by } S_{ALG}] &= \prod_{S_u} (1 - x_s^*), \text{ since } X_s \text{ independent} \\ &\leq \prod_{S_u} e^{-x_s^*}, \text{ since } 1 + y \leq e^y \text{ for all } y \in \mathbb{R} \\ &= e^{-\sum_{S_u} x_s^*}, \text{ because of algebra} \\ &\leq \frac{1}{e} \approx 0.37, \text{ since } \sum_{S_u} x_s^* \geq 1 \end{aligned}$$

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Highly unlikely S_{ALG} will cover all U, but each individual element has good likelihood of being covered.

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2 - Repeat step 1 T-times, while adding sets to S_{ALG}.

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$$E[ALG] = E\left[\sum_{s\in S_{ALG}} X_s\right]$$
$$\leq E\left[\sum_{t\leq T} \sum_{s\in S_{ALG_t}} X_s\right]$$, since multiple iterations may select s

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Define random variable $X_s = \begin{cases} 1, & s \in S_{ALG} \\ 0, & \text{otherwise} \end{cases}$

$E[ALG] = E\left[\sum_{s \in S_{ALG}} X_s\right] = E\left[\sum_{s \in S} X_s\right]$, since the rest of the $X_s = 0$

$= \sum_{s \in S} E[X_s]$

$= \sum_{s \in S} x_s^*$, since x_s^* is probability $X_s = 1$

$= OPT_{LP} \leq OPT$

Set Cover ILP

Objective: $\min \sum_s x_s$
Subject to: $\sum_{s:\, u \in s} x_s \geq 1$, for each $u \in U$
$0 \leq x_s \leq 1$, for each set s

+

1 - Add set s to our subset S_{ALG} with probability of x_s.

2 - Repeat step 1 T-times, while adding sets to S_{ALG}.

Recall: x_s^* = optimal solutions to LP relaxation

$$X_s = \begin{cases} 1, & s \in S_{ALG} \\ 0, & \text{otherwise} \end{cases}$$

Suppose $T = \ln 4n$, where $|U| = n$.

1. What is the size of the solution?

$$E[ALG] = E\left[\sum_{s \in S_{ALG}} X_s\right]$$

$$\leq E\left[\sum_{t \leq T} \sum_{s \in S_{ALG_t}} X_s\right]$$, since multiple iterations may select s

$$= T \sum_{s \in S} x_s^*$$, by previous bound $E\left[\sum_{s \in S_{ALG}} X_s\right] = \sum_{s \in S} x_s^*$

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We want to show that:

$$ALG \leq 4 \ln 4n \;\; OPT$$

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What is the chance that $ALG \leq 4 \ln 4n \; OPT$ does not happen?

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Thus, $\Pr[ALG > 4 \ln 4n\, OPT]$

What is the chance that $ALG \leq 4 \ln 4n\ \ OPT$ does not happen?

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$= T \sum_{s \in S} x_s^*$, by previous bound $E\left[\sum_{s \in S_{ALG}} X_s\right] = \sum_{s \in S} x_s^*$

Thus, $\Pr[ALG > 4 \ln 4n\, OPT] \leq \Pr[ALG > 4T \sum_{s \in S} x_s^*]$

Probability some event happens increases as threshold decreases.

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Thus, $\Pr[ALG > 4 \ln 4n \, OPT] \leq \Pr[ALG > 4T \sum_{s \in S} x_s^*]$

$\boldsymbol{OPT \geq OPT_{LP} = \sum_{s \in S} x_s^*}$

Probability some event happens increases as threshold decreases.

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Suppose $T = \ln 4n$, where $|U| = n$.

Markov's Inequality:
$\Pr[X \geq a] \leq \frac{E[X]}{a}$

1. What is the size of the solution?

$$E[ALG] = E\left[\sum_{s \in S_{ALG}} X_s\right]$$
$$\leq E\left[\sum_{t \leq T} \sum_{s \in S_{ALG_t}} X_s\right] \text{, since multiple iterations may select } s$$
$$= T \sum_{s \in S} x_s^* \text{, by previous bound } E\left[\sum_{s \in S_{ALG}} X_s\right] = \sum_{s \in S} x_s^*$$

Thus, $\Pr[ALG > 4 \ln 4n\, OPT] \leq \Pr[ALG > 4T \sum_{s \in S} x_s^*]$

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Thus, $\Pr[ALG > 4 \ln 4n\, OPT] \leq \Pr[ALG > 4T \sum_{s \in S} x_s^*]$

$$\leq \frac{E[ALG]}{4T \sum_{s \in S} x_s^*}$$

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Thus, $\Pr[ALG > 4 \ln 4n\, OPT] \leq \Pr[ALG > 4T \sum_{s \in S} x_s^*]$
$$\leq \frac{E[ALG]}{4T \sum_{s \in S} x_s^*} = \frac{T \sum_{s \in S} x_s^*}{4T \sum_{s \in S} x_s^*}$$

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Thus, $\Pr[ALG > 4 \ln 4n\, OPT] \leq \Pr[ALG > 4T \sum_{s \in S} x_s^*]$
$$\leq \frac{E[ALG]}{4T \sum_{s \in S} x_s^*} = \frac{T \sum_{s \in S} x_s^*}{4T \sum_{s \in S} x_s^*} = \frac{1}{4}$$

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Suppose $T = \ln 4n$, where $|U| = n$.

2. What is the probability solution is valid?

   Let S_u be sets of S that contain element u.

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Suppose $T = \ln 4n$, where $|U| = n$.

2. What is the probability solution is valid?

   Let S_u be sets of S that contain element u.

   $\Pr[u \text{ not covered by } S_{ALG}] = \prod_{t \leq T} \Pr[u \text{ not covered by } S_{ALG_t}]$

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Recall: x_s^* = optimal solutions to LP relaxation

$$X_s = \begin{cases} 1, & s \in S_{ALG} \\ 0, & \text{otherwise} \end{cases}$$

2. What is the probability solution is valid?

Let S_u be sets of S that contain element u.

u is covered by $S_{ALG} \Leftrightarrow \sum_{S_u} X_s \geq 1$.

$\Pr[u \text{ not covered by } S_{ALG}] = \prod_{S_u}(1 - x_s^*)$, since X_s independent

$\leq \prod_{S_u} e^{-x_s^*}$, since $1 + y \leq e^y$ for all $y \in \mathbb{R}$

$= e^{-\sum_{S_u} x_s^*}$, because of algebra

$\leq \frac{1}{e} \approx 0.37$, since $\sum_{S_u} x_s^* \geq 1$

Highly unlikely S_{ALG} will cover all U, but each individual element has good likelihood of being covered.

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   $$= \frac{1}{e^T}$$

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Thus, $\Pr[S_{ALG}$ is not a cover$]$

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Thus, $\Pr[S_{ALG} \text{ is not a cover}] = \Pr[\cup_u (u \text{ not covered by } S_{ALG})]$

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Union Bound:
$\Pr[\cup_i A_i] \leq \sum_i \Pr[A_i]$

Thus, $\Pr[S_{ALG} \text{ is not a cover}] = \Pr[\cup_u (u \text{ not covered by } S_{ALG})]$

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$\Pr[\bigcup_i A_i] \leq \sum_i \Pr[A_i]$

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$$\leq \sum_u \Pr[u \text{ not covered by } S_{ALG}], \text{ by union bound}$$
$$\leq \sum_u \frac{1}{4n} = n \frac{1}{4n} = \frac{1}{4}$$

Set Cover ILP

$\mathbf{Pr} = \frac{\mathbf{3}}{\mathbf{4}}$ $\mathbf{Pr} = \frac{\mathbf{3}}{\mathbf{4}}$

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Ten times? $1 - \left(\frac{7}{16}\right)^{10} \approx 0.9997$

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High likelihood of "good" solution with guaranteed polynomial time algorithm.

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