(F)PTAS
CSCI 532

Knapsack

Knapsack: Given a set of n items with values $v_1, \ldots, v_n$ and weights $w_1, \ldots, w_n$, select the most valuable combination with total weight $\leq W$.

Example:

$W = 11$

Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

$\{1, 2, 5\}$: weight = 10, value = 35

$\{3, 4\}$: weight = 11, value = 40

$\{4, 5\}$: weight = 13, value = 50

Knapsack – Dynamic Programming

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Algorithm:

Is there optimal substructure?

I.e. If I have an optimal solution to some instance, does that imply an optimal solution to a different instance?

Yes, removing some item must give an optimal selection for the remaining weight.

E.g. If {item$_1$, item$_3$, item$_7$} is optimal for weight of 10, and item$_3$ weighs 3, {item$_1$, item$_7$} ***must*** be optimal for a weight of 7.

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Define $\text{opt}(i, w) =$ maximum value achievable for items $\{1, \dots, i\}$ and knapsack of capacity w.

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What now?

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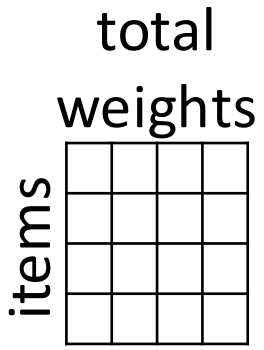
What now?

- For every single weight $\leq W$ and i, calculate $\text{opt}(i, w)$.



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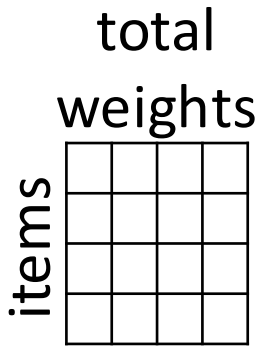
- For every single weight $\leq W$ and i, calculate $\text{opt}(i, w)$.
- No longer polynomial.



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- For every single weight $\leq W$ and i, calculate $\text{opt}(i, w)$.
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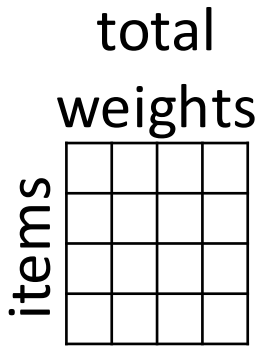
- Reducing search space of weights will compromise weight accuracy. Not allowed!



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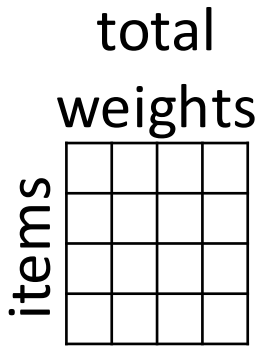

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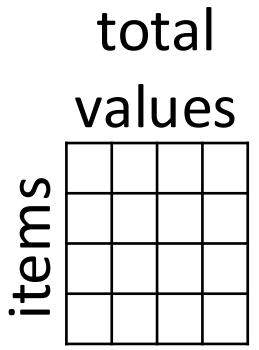

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Define $\text{opt}(i, V) =$ minimum weight of subset of items $\{1, \dots, i\}$ that gives value at least V.

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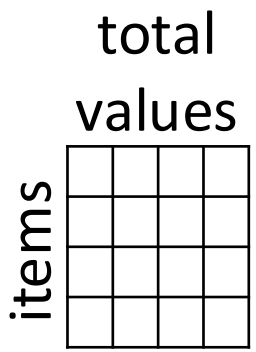

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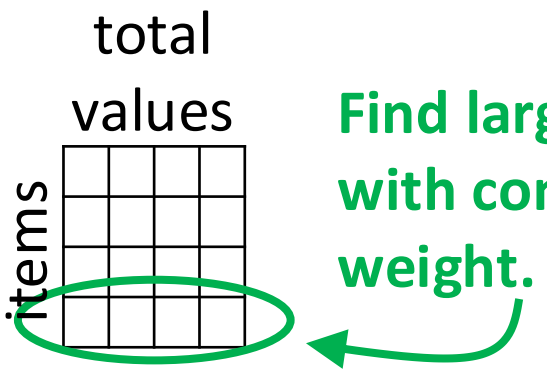
Define $\text{opt}(i, V) =$ minimum weight of subset of items $\{1, \dots, i\}$ that gives value at least V.

To find the optimal solution, find largest V such that $\text{opt}(n, V) \leq W$.



Find largest value with compatible weight.

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If item i is in the optimal solution, removing it decreases the value by v_i and weight by w_i. What remains must be the minimum weight whose value is at least $V - v_i$.

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Define $\text{opt}(i, V) =$ minimum weight of subset of items $\{1, \dots, i\}$ that gives value at least V.

$$\text{opt}(i, V) = \begin{cases} 0 & \text{if } V = 0 \\ \infty & \text{if } i = 0 \text{ and } V > 0 \\ \min \begin{pmatrix} \text{opt}(i-1, V), \\ \text{opt}(i-1, V - v_i) + w_i \end{pmatrix} & \text{Otherwise} \end{cases}$$

Knapsack – Algorithm

Suppose values are integers.

Knapsack: Given a set of n items with values $v_1, \dots, v_n$ and weights $w_1, \dots, w_n$, select the most valuable combination with total weight $\leq W$.

Algorithm:

Define $\text{opt}(i, V) =$ minimum weight of subset of items $\{1, \dots, i\}$ that gives value at least V.

1. For $0 \leq i \leq n$ and $0 \leq V \leq$? , compute $\text{opt}(i, V)$.

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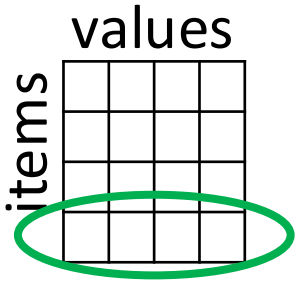
2. Return $V^* = \max \{V : \text{opt}(i, V) \leq W\}$.



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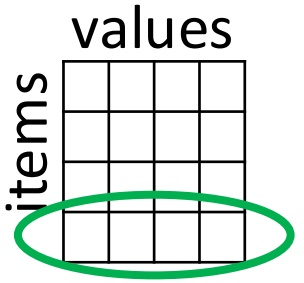
Running Time = ?



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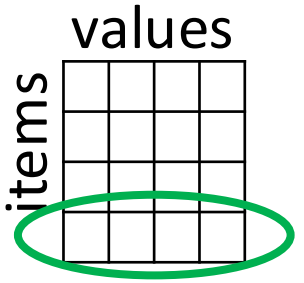
Running Time = $O(n^2 \max_i v_i)$



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Optimal algorithm that does not run in polynomial time WRT input size.

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v_i
- 100000
- 200000
- 500000

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v_i
- 100000 — 1
- 200000 → 2
- 500000 — 5

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Running Time = $O(n^2 \max_i v_i)$

v_i
- 163882 — 1
- 254301 → 2
- 582242 — 5

Knapsack – Algorithm

Knapsack: Given a set of n items with values $v_1, \dots, v_n$ and weights $w_1, \dots, w_n$, select the most valuable combination with total weight $\leq W$.

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Running Time = $O(n^2 \max_i v_i)$

Does this change the optimal value all that much?

v_i
- 163882 — 1
- 254301 → 2
- 582242 — 5

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Does this change the optimal value all that much?

v_i
163882 → 163
254301 → 254
582242 → 582

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v_i	163882	16388
	254301 →	25430
	582242	58224

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Running Time = $O(n^2 \max_i v_i)$

There seems to be a tradeoff that we can control between accuracy and running time.

v_i		
163882		16388
254301	→	25430
582242		58224

Knapsack – Algorithm

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Algorithm:

1. Let $v_{\max} = \max_i v_i$
2. For each i, let $v_i' = \left\lfloor v_i \frac{n}{\varepsilon v_{\max}} \right\rfloor$

User-supplied approximation factor $\varepsilon > 0$.

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$$\begin{matrix} 726489 \\ 136212 \\ 384167 \end{matrix} \xrightarrow{\varepsilon = 0.1} \begin{matrix} 30 \\ 5 \\ 15 \end{matrix}$$

$$\begin{matrix} 726489 \\ 136212 \\ 384167 \end{matrix} \xrightarrow{\varepsilon = 0.001} \begin{matrix} 3000 \\ 562 \\ 1586 \end{matrix}$$

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Running Time = $O(n^2 v'_{\max})$

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Running Time = $O(n^2 v'_{\max}) \in O\left(n^2 \left\lfloor v_{\max} \frac{n}{\varepsilon v_{\max}} \right\rfloor\right)$

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2. For each i, let $v_i' = \left\lfloor v_i \frac{n}{\varepsilon v_{\max}} \right\rfloor$
3. Run dynamic programming algorithm using $\{v_i'\}, \{w_i\}, W$

Running Time = $O(n^2 v'_{\max}) \in O\left(n^2 \left\lfloor v_{\max} \frac{n}{\varepsilon v_{\max}} \right\rfloor\right) \in O\left(\frac{n^3}{\varepsilon}\right)$

Knapsack – Algorithm

Knapsack: Given a set of n items with values $v_1, \ldots, v_n$ and weights $w_1, \ldots, w_n$, select the most valuable combination with total weight $\leq W$.

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How does scaling values impact cost?

Knapsack – Performance

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Need to relate $\sum_{i \in S_{\text{OPT}}} v_i$ relate to $\sum_{i \in S_{\text{ALG}}} v_i$?

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But the algorithm operates on v_i'.

Need to relate $\sum_{i \in S_{\text{OPT}}} v_i$ to $\sum_{i \in S_{\text{OPT}}} v_i'$ to $\sum_{i \in S_{\text{ALG}}} v_i'$ to $\sum_{i \in S_{\text{ALG}}} v_i$

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S_{ALG} is optimal for v_i'

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$\geq \text{OPT} - \varepsilon\,\text{OPT}$, because $\text{OPT} \geq v_{\max}$ (discard overweight items)

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$= (1 - \varepsilon)\ \text{OPT}$

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So, $\text{ALG} \geq (1 - \varepsilon)\ \text{OPT}$

Knapsack

Knapsack: Given a set of n items with values $v_1, \ldots, v_n$ and weights $w_1, \ldots, w_n$, select the most valuable combination with total weight $\leq W$.

Performance Guarantee: ALG $\geq (1 - \varepsilon)$ OPT

Running Time: $O\left(\frac{n^3}{\varepsilon}\right)$

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We can solve Knapsack instances arbitrarily close to optimal in polynomial time!!

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Fully Polynomial-Time Approximation Scheme (FPTAS)

We can solve Knapsack instances arbitrarily close to optimal in polynomial time!!

Approximability Hierarchy

PTAS: Running time polynomial in input size.

FPTAS: Running time polynomial in input size and ε.