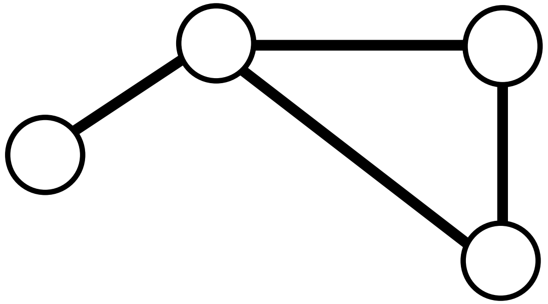
Minimum Spanning Trees
CSCI 532

Minimum Spanning Tree (MST)



Given a connected graph, a subset of edges is a…

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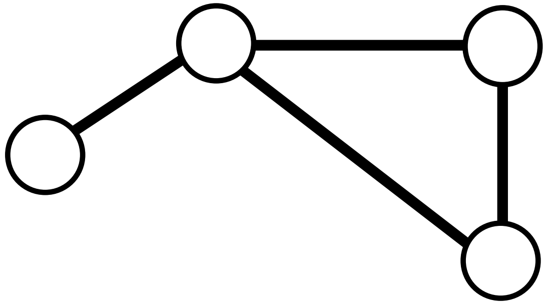
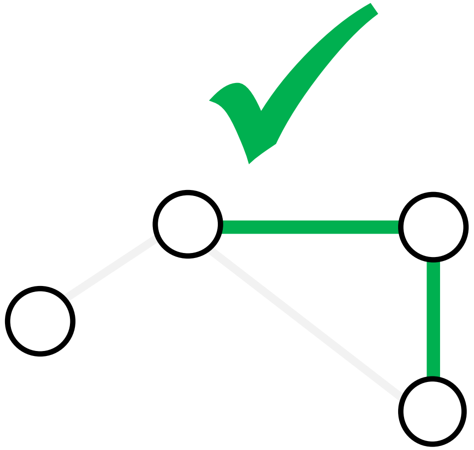
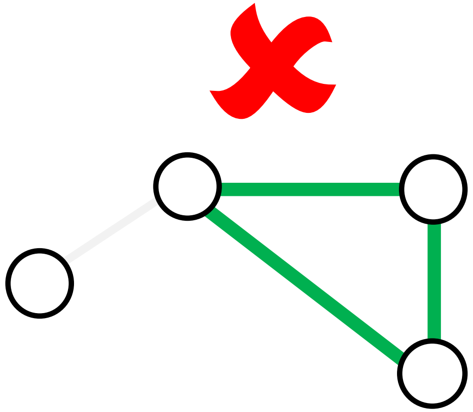


Given a connected graph, a subset of edges is a…

Tree if it is connected and acyclic.





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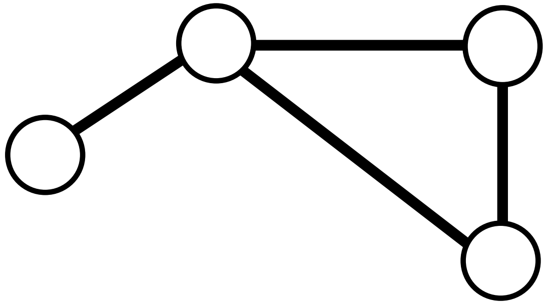

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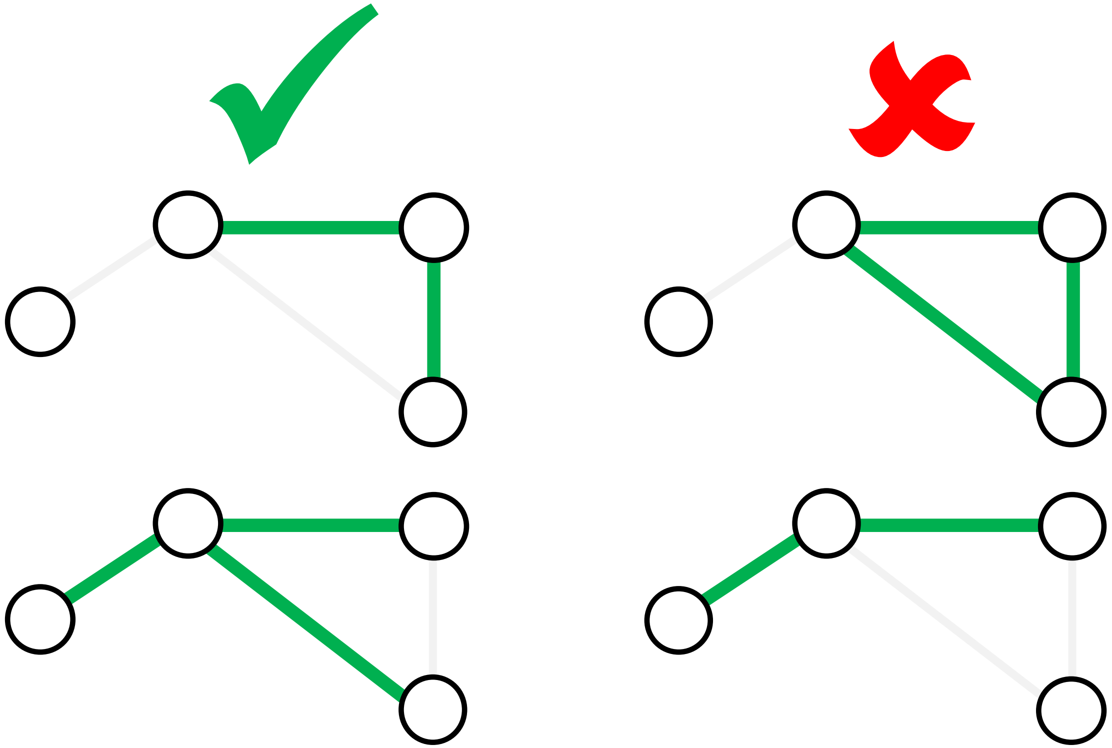

Tree if it is connected and acyclic.

Spanning tree if it is a tree and includes all vertices in the graph.

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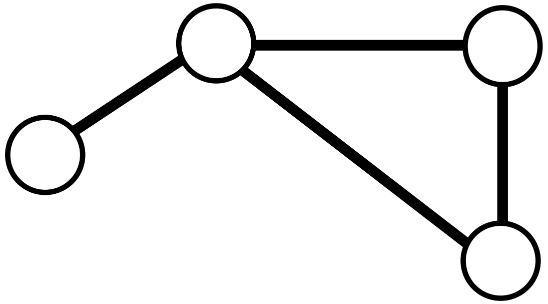

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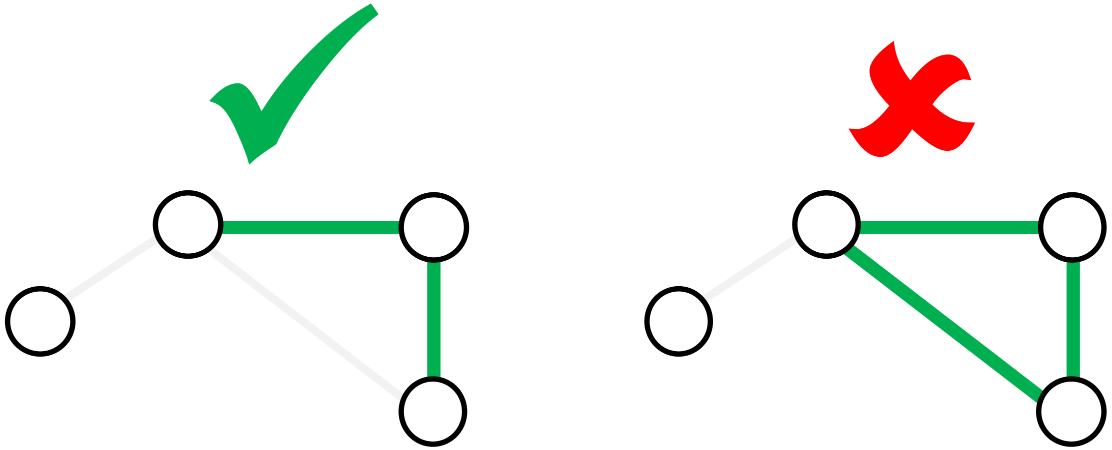

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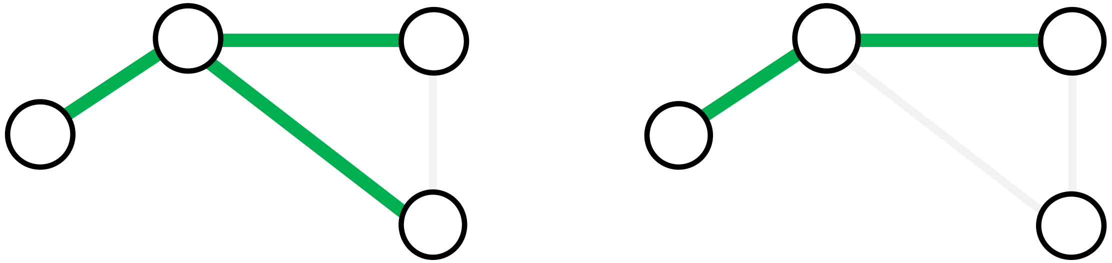
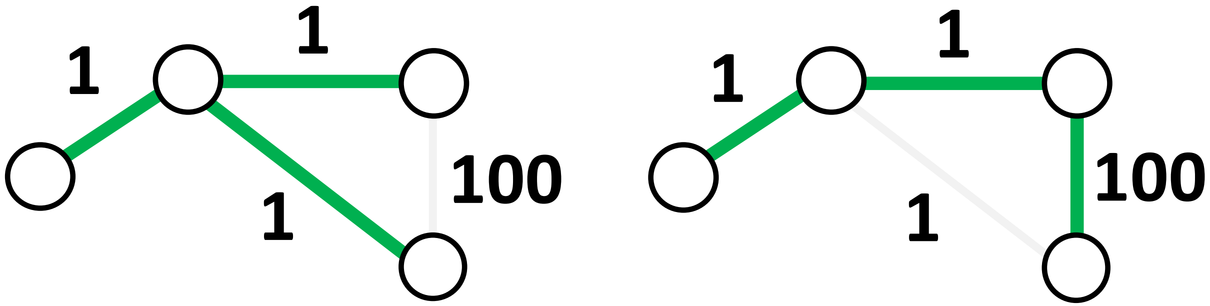
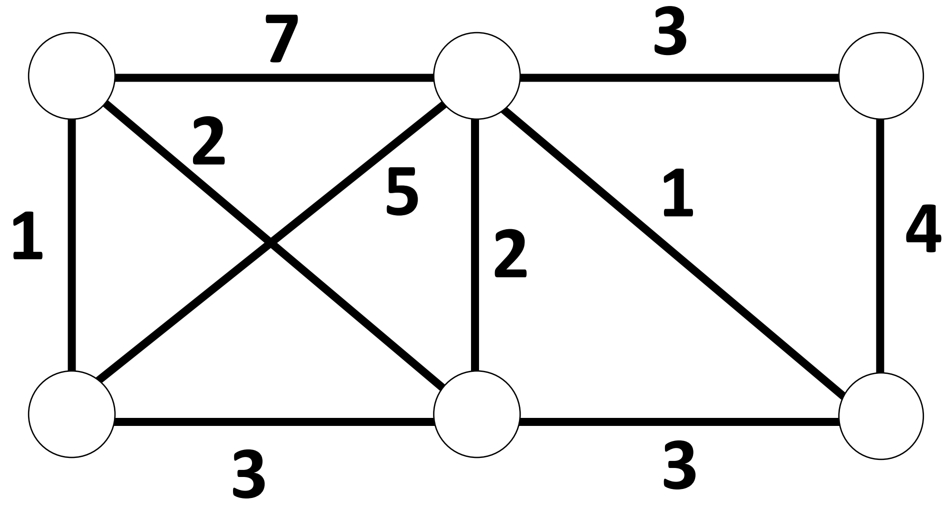


Minimum spanning tree if it is a spanning tree whose sum of edge costs is the minimum possible value.

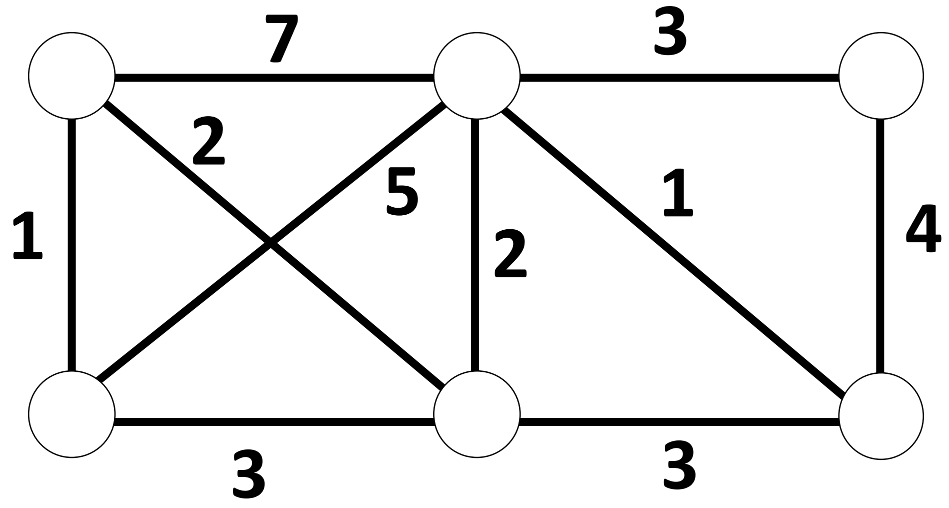


MST Problem



Goal: Given a connected, edge weighted graph, find its Minimum Spanning Tree.

MST Problem

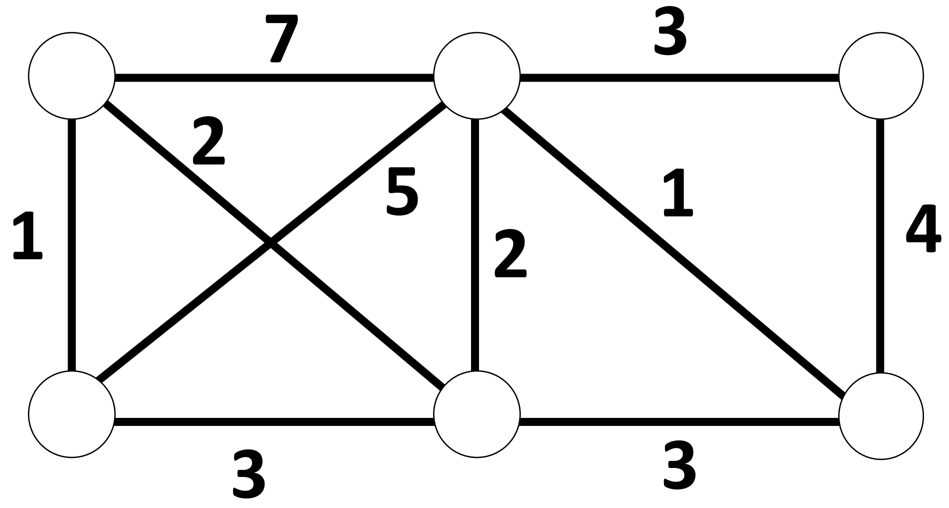


Greedy Algorithms:

- Make the choice that best helps some objective.
- Do not look ahead, plan, or revisit past decisions.
- Hope that optimal local choices lead to optimal global solutions.

Kruskal's MST Algorithm

Algorithm: Add the edge with smallest weight, that does not create a cycle.



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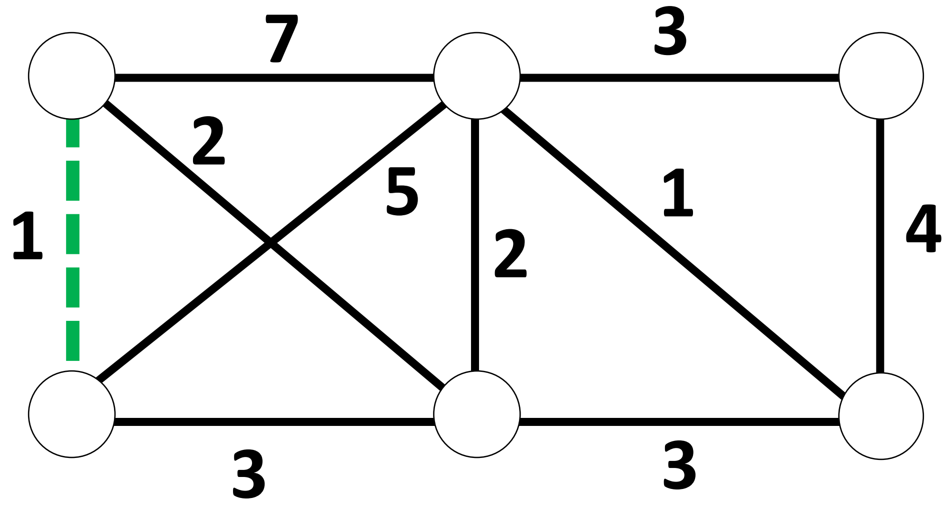

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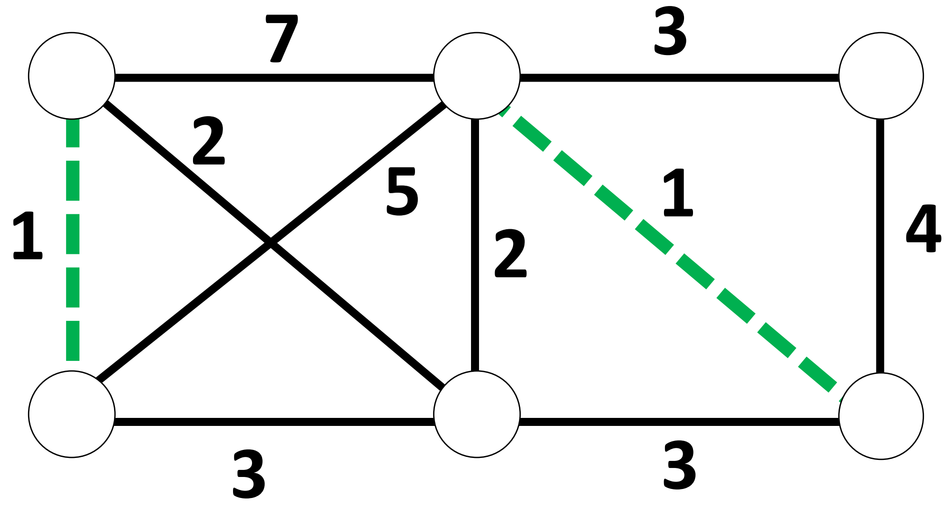

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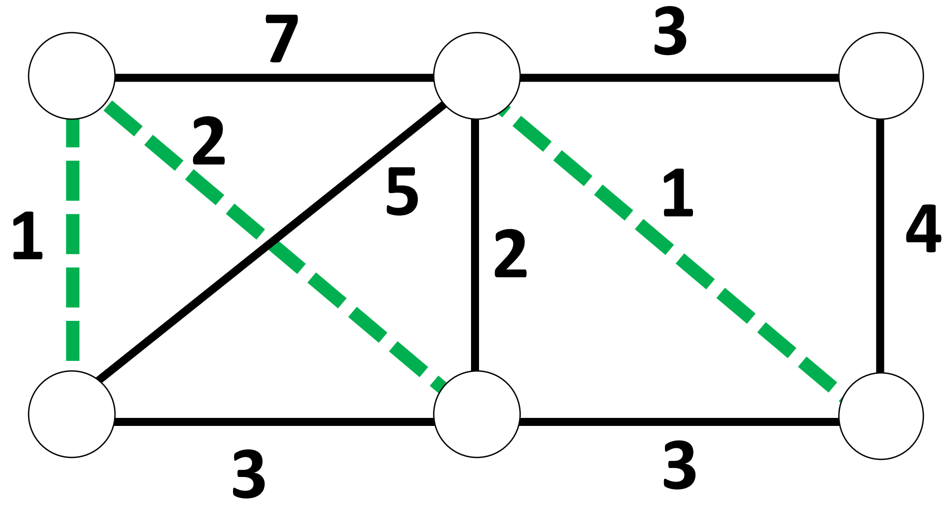

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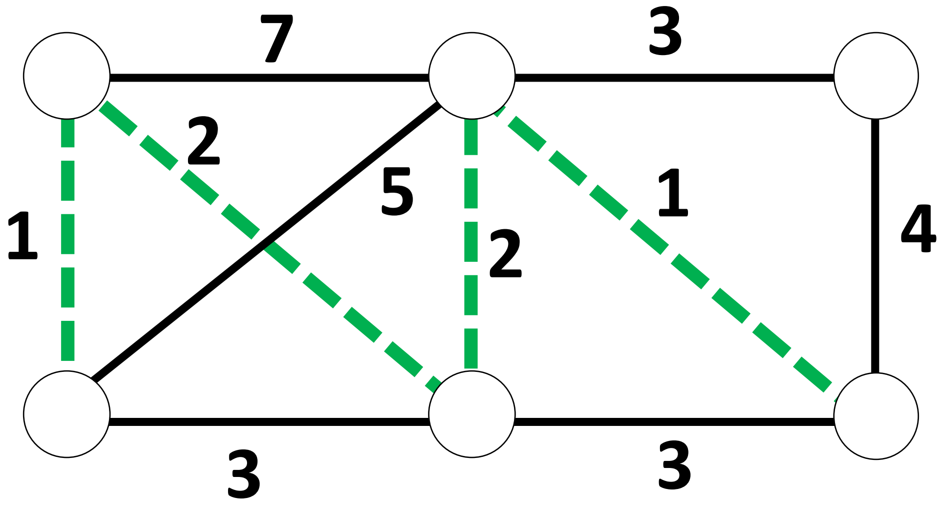

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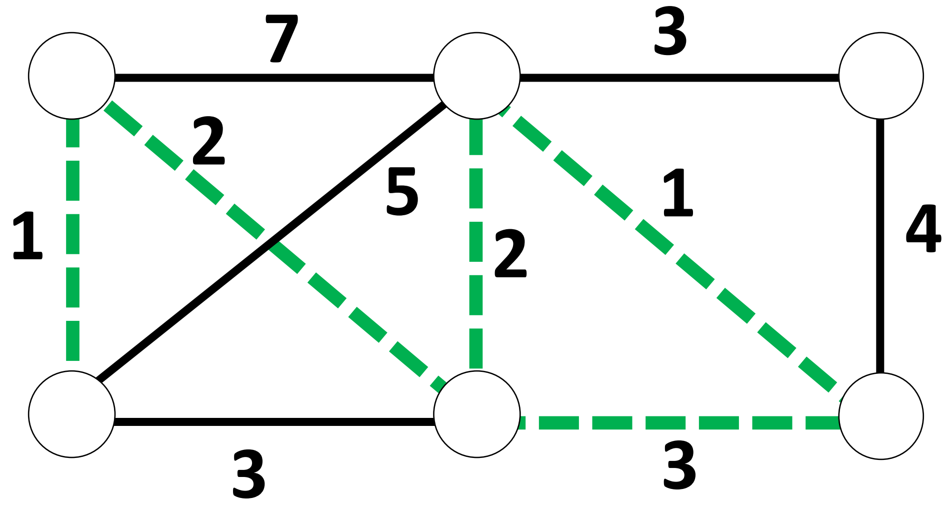

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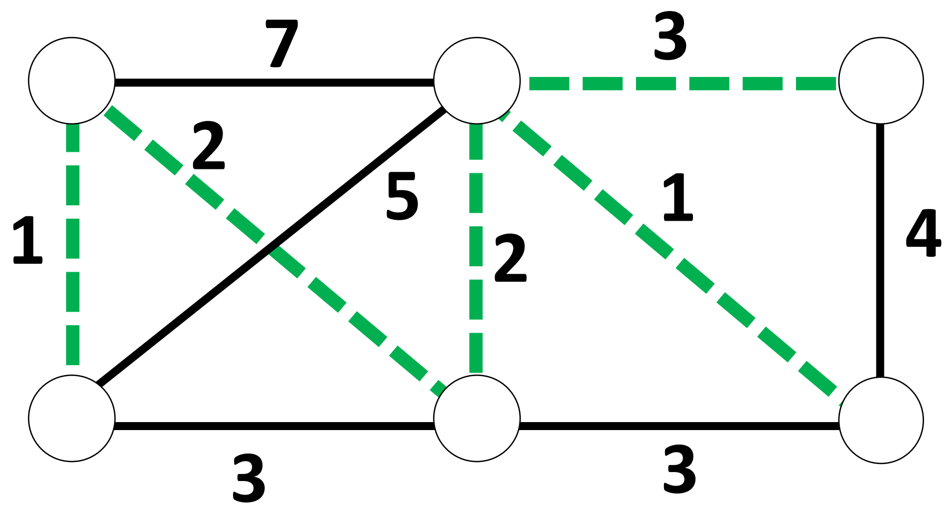

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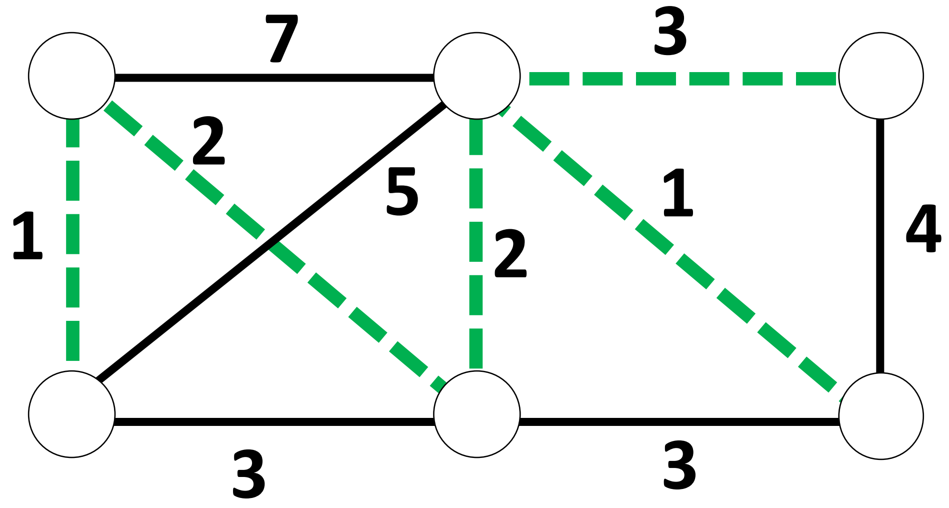

What are some questions we may have about the algorithm?

1. Is the solution valid? (Does it actually find a spanning tree?)
2. What is the running time?
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Kruskal’s MST Algorithm

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Proof of validity: ?

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What do we need to show?

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Algorithm: Add the edge with smallest weight, that does not create a cycle.

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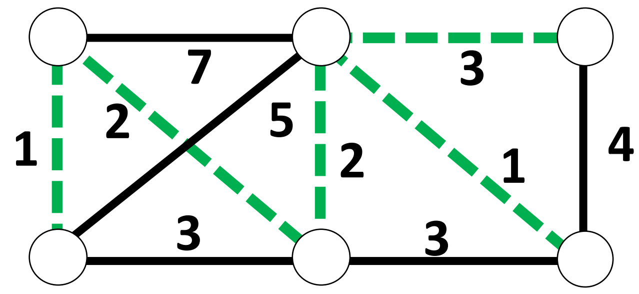
T is a tree because it is connected (otherwise we could have added more edges without creating cycles) and there are no cycles.



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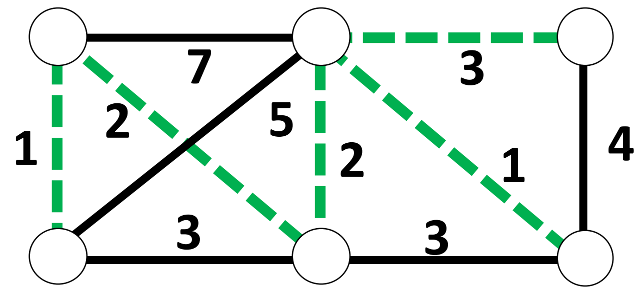
T spans G because if it did not, we could have added more edges to connected unreached nodes without creating cycles.



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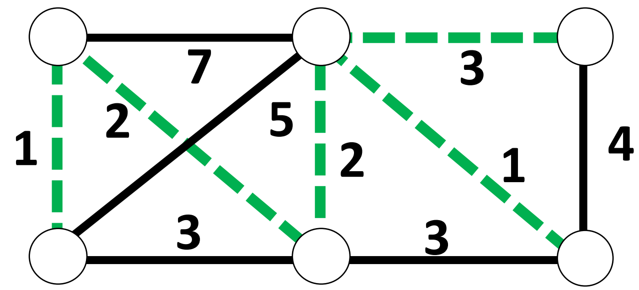
$\therefore$ T is a spanning tree of G



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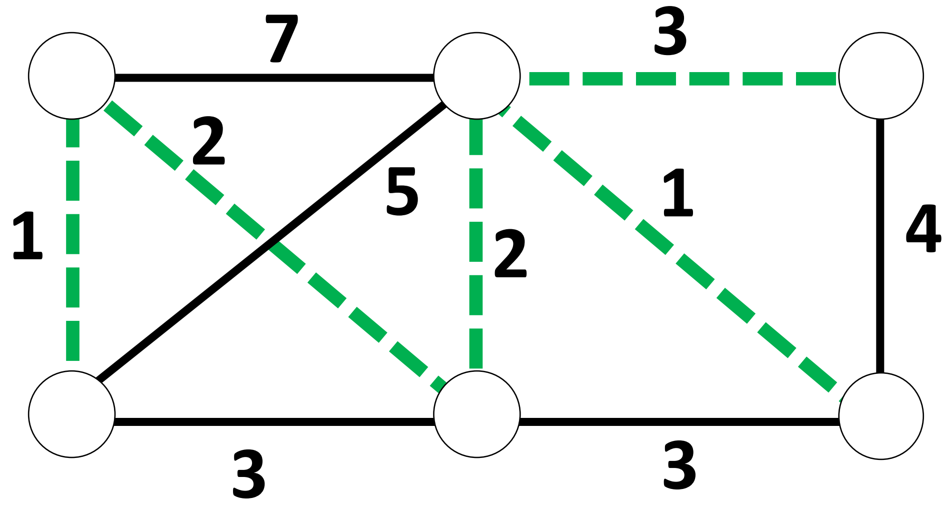

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Kruskal’s MST Algorithm

Algorithm: Add the edge with smallest weight, that does not create a cycle.

Running Time:

Kruskal's MST Algorithm

<u>Algorithm:</u> Add the edge with smallest weight, that does not create a cycle.

<u>Running Time:</u>

```
findMST(G=(V,E)) {
  T = Ø
  sort(E) //smallest to largest weight
  for (e in E) {
    if (T U {e} is acyclic) {
      T = T U {e}
    }
  }
  return T
}
```

Kruskal's MST Algorithm

Algorithm: Add the edge with smallest weight, that does not create a cycle.

Running Time:

```
findMST(G=(V,E)) {
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← $O(|E| \log(|E|))$

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findMST(G=(V,E)) {
  T = ∅
  sort(E) //smallest to largest weight   ← O(|E| log(|E|))
  for (e in E) {   ← O(|E|)
    if (T ∪ {e} is acyclic) {   ← O(|V| + |E|) using BFS
      T = T ∪ {e}
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sort(E) ← $O(|E|\log(|E|))$

for (e in E) ← $O(|E|)$

if (T ∪ {e} is acyclic) ← $O(|V| + |E|)$ **using BFS**

Running time

$$\in O\big(|E|\log(|E|) + |E|(|V| + |E|)\big)$$

$$\in O\big(|E|^2 + |E||V|\big)$$

Kruskal’s MST Algorithm

Algorithm: Add the edge with smallest weight, that does not create a cycle.

Running Time:

Can be improved to $O(1)$, thus $O(|E| \log(|E|))$ overall

```
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- sort(E): $O(|E| \log(|E|))$
- for (e in E): $O(|E|)$
- if (T ∪ {e} is acyclic): $O(|V| + |E|)$ using BFS

Running time

$$\in O\big(|E| \log(|E|) + |E|(|V| + |E|)\big)$$
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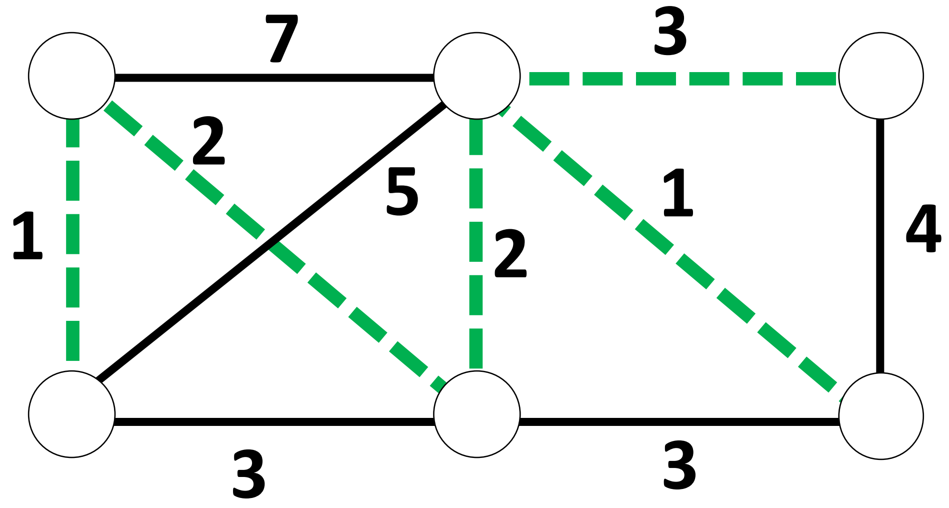

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Algorithm: Add the edge with smallest weight, that does not create a cycle.

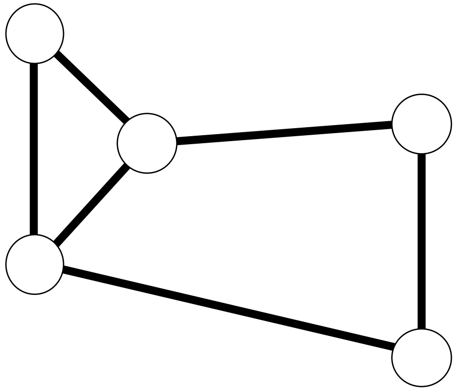
Proof of optimality: T is an MST, because???

MST Cut Property

Assume unique edge costs.

Lemma: Suppose that S is a subset of nodes from $G = (V, E)$. Then, the cheapest edge e between S and $V \backslash S$ is part of the MST.

Proof:



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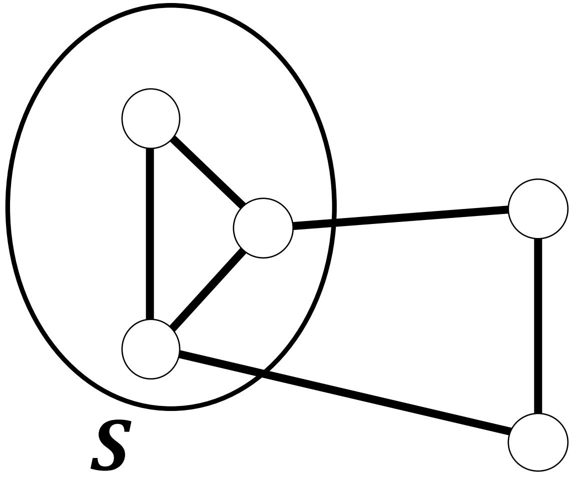

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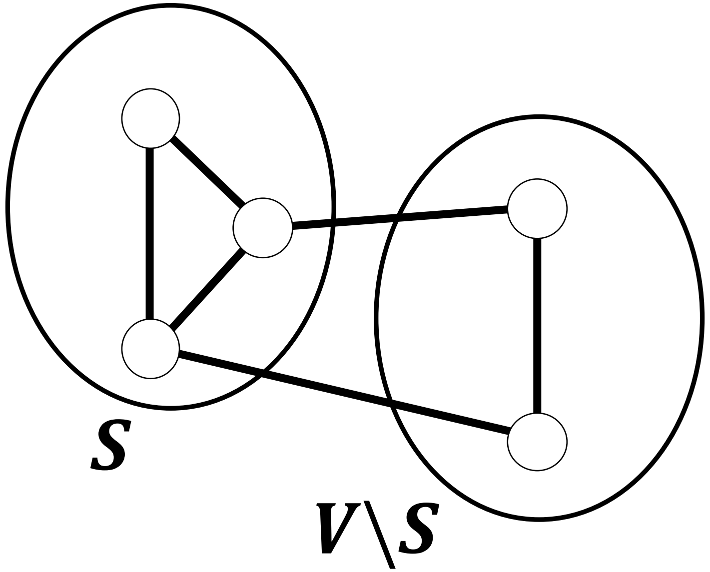

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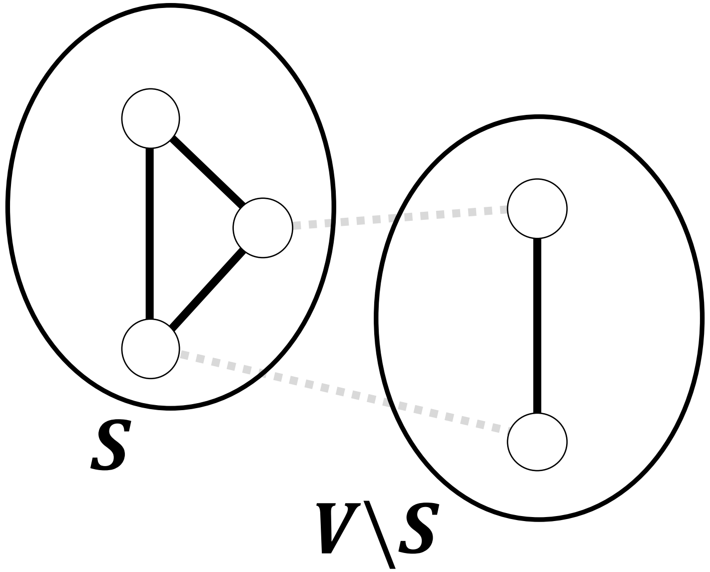
Proof: Any MST of G must include some edge between S and $V\backslash S$ (otherwise it would not be a spanning tree).



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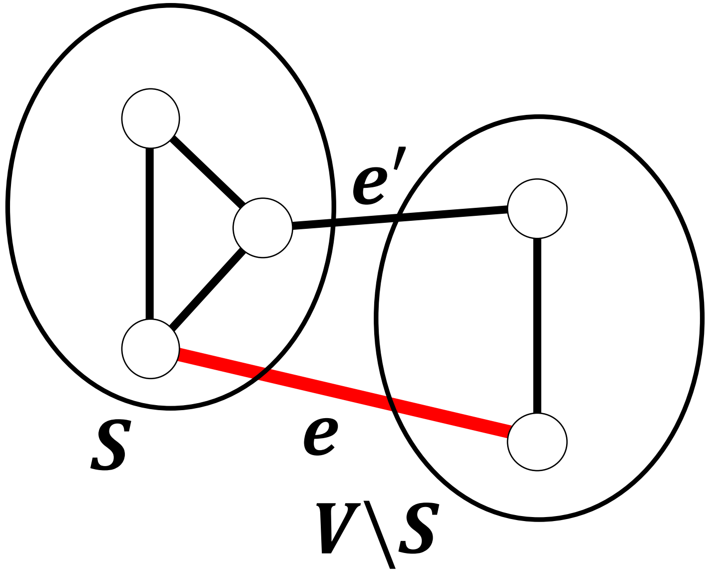

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Lemma: Suppose that S is a subset of nodes from $G = (V, E)$. Then, the cheapest edge e between S and $V \setminus S$ is part of the MST.

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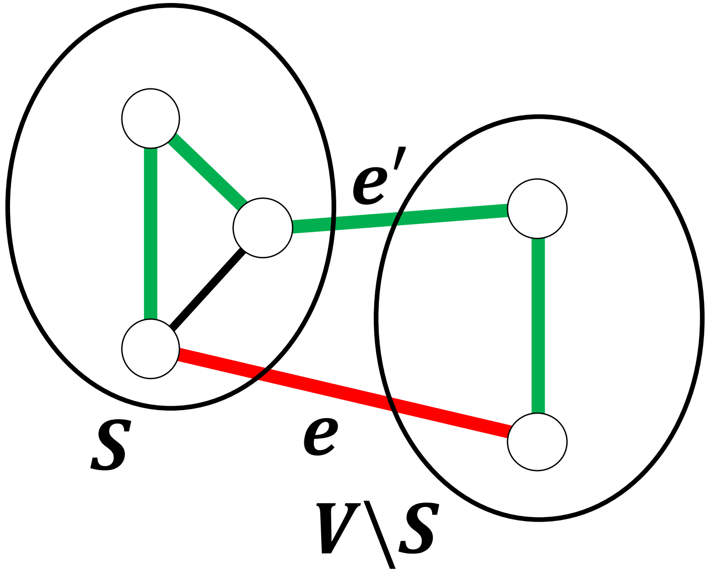
Suppose $\boldsymbol{T}$ is the MST that does not include $\boldsymbol{e}$.



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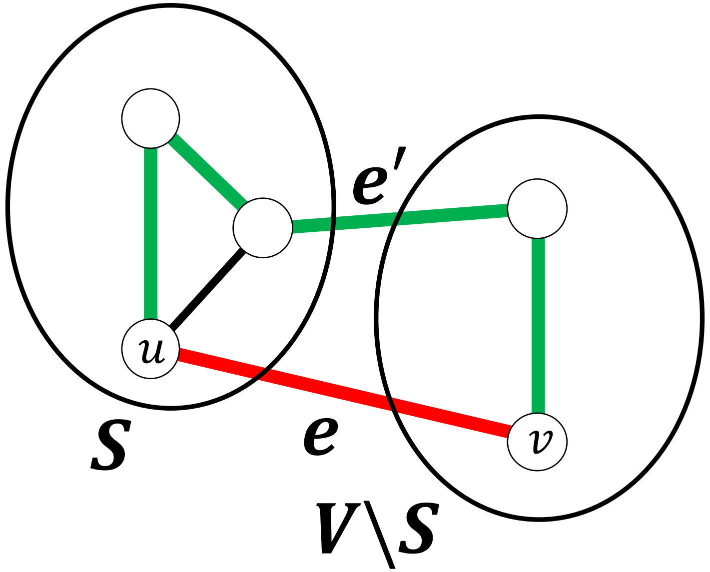
Let $\boldsymbol{e}$ be the cheapest edge between S and $V \backslash S$.

Suppose $\boldsymbol{T}$ is the MST that does not include $\boldsymbol{e}$. Then:

1. $\boldsymbol{T} \cup \{\boldsymbol{e}\}$ must have a cycle. Because?



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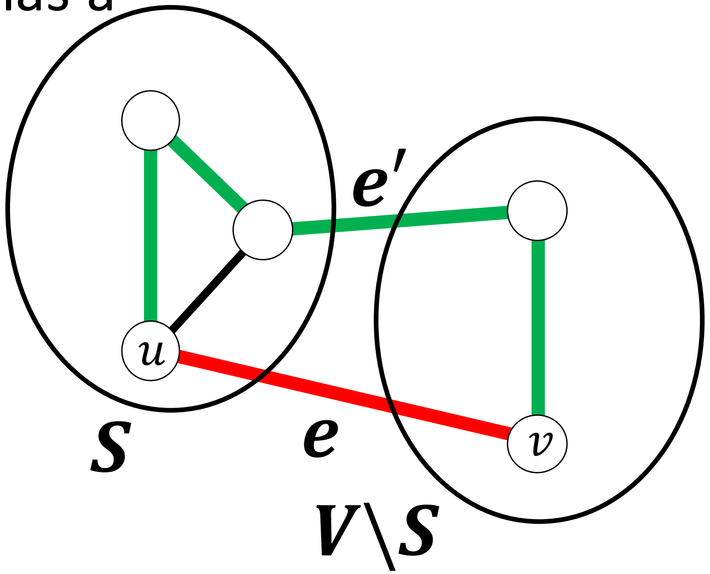

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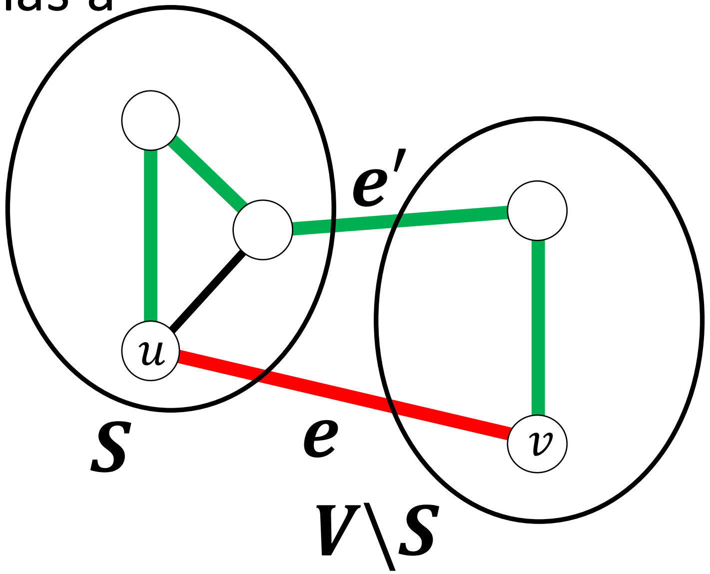

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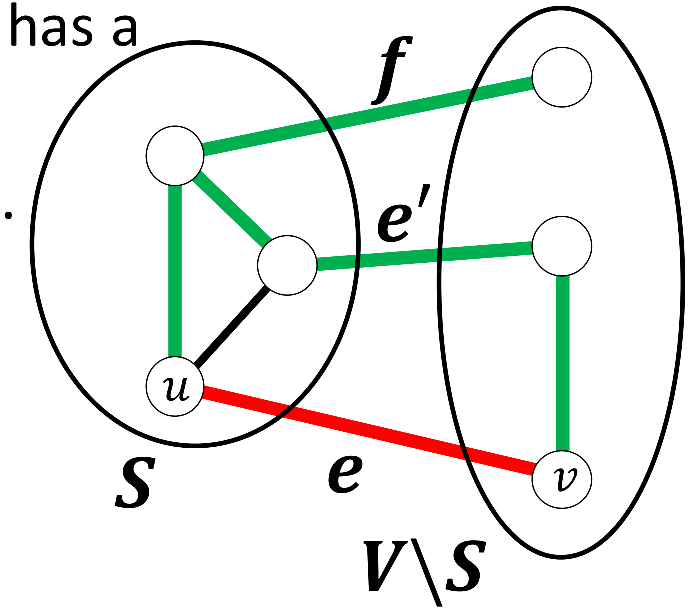

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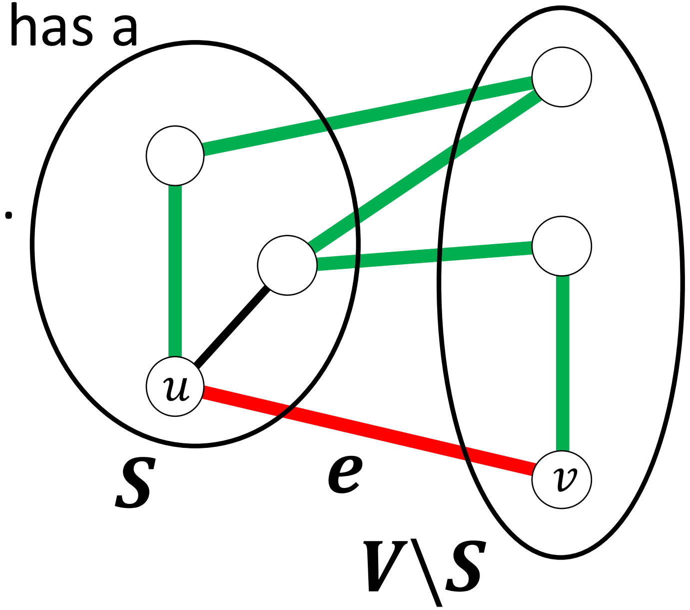

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(Which one doesn't matter.)

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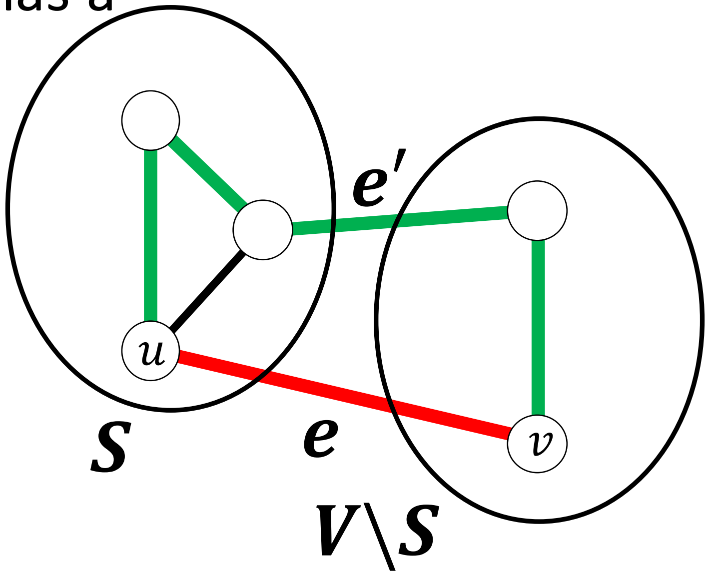

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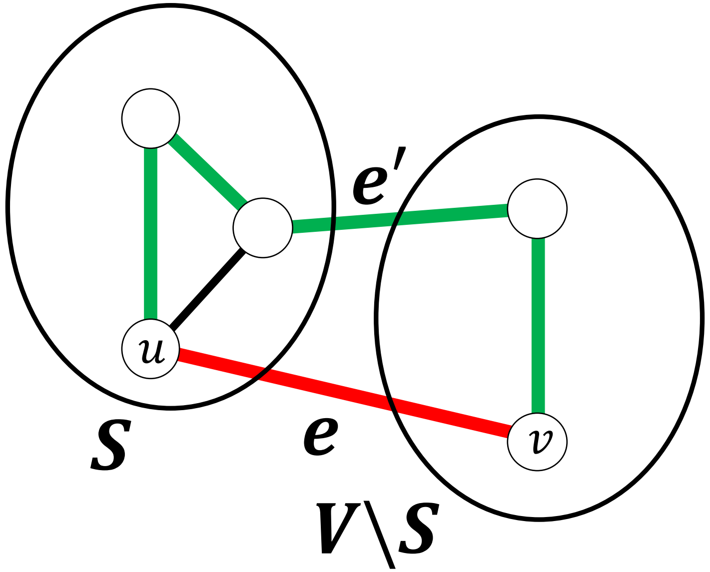

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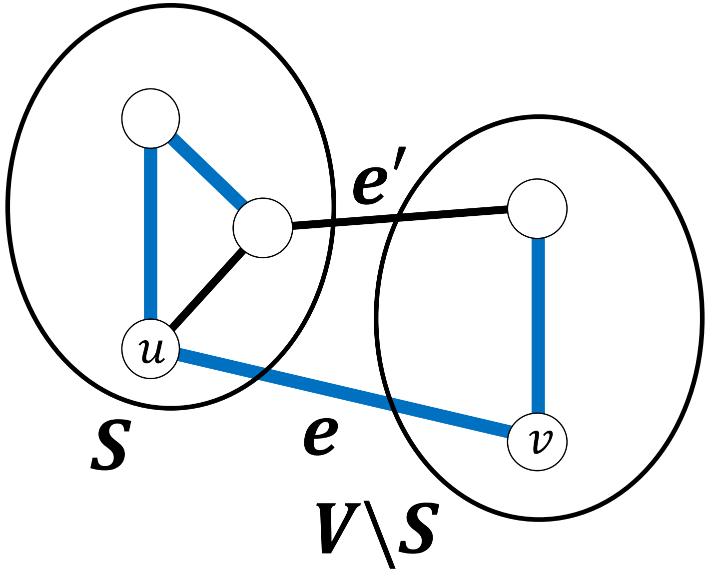

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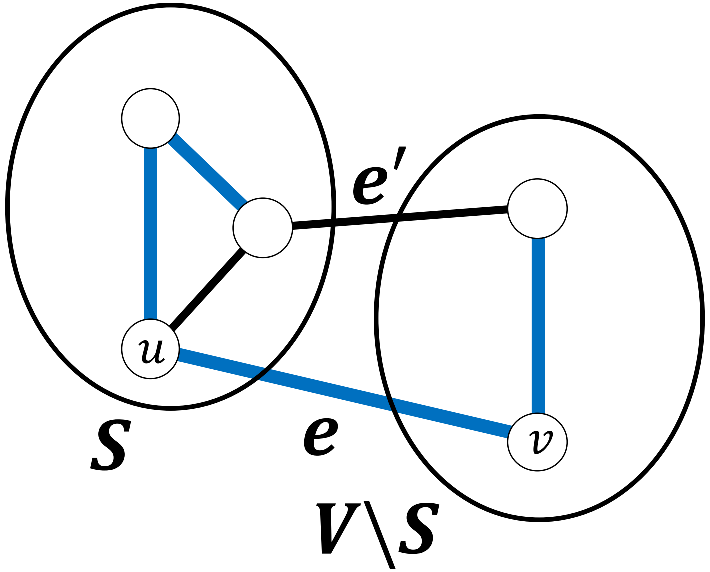

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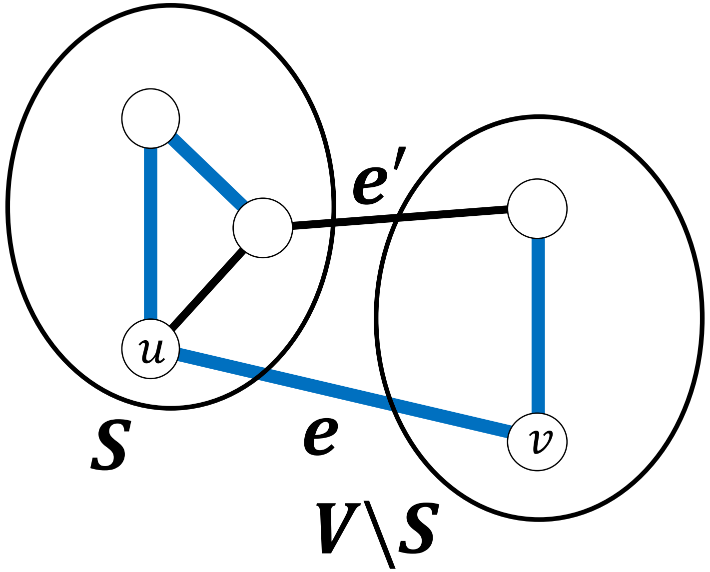

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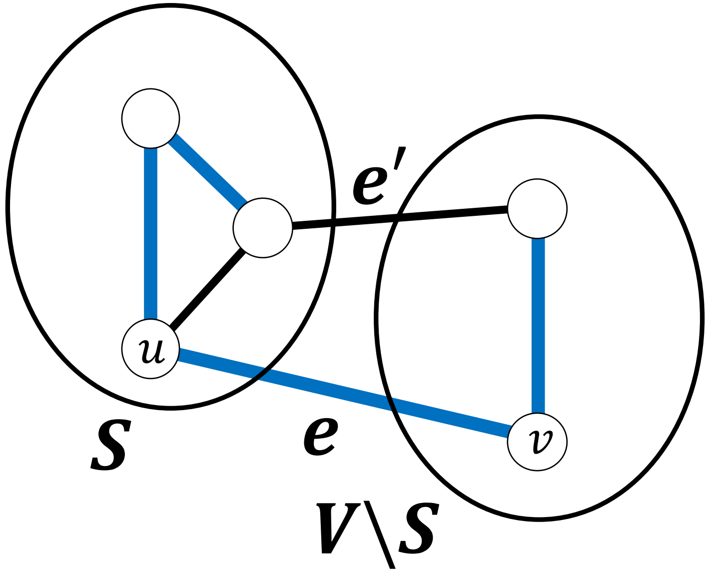

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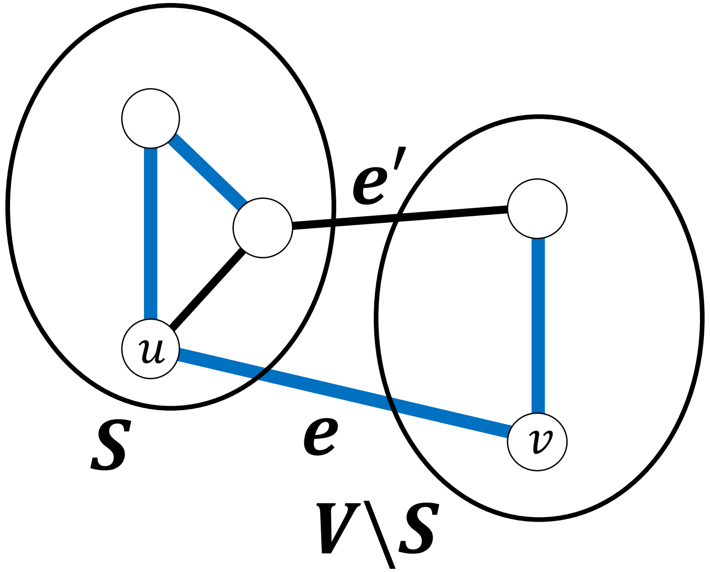

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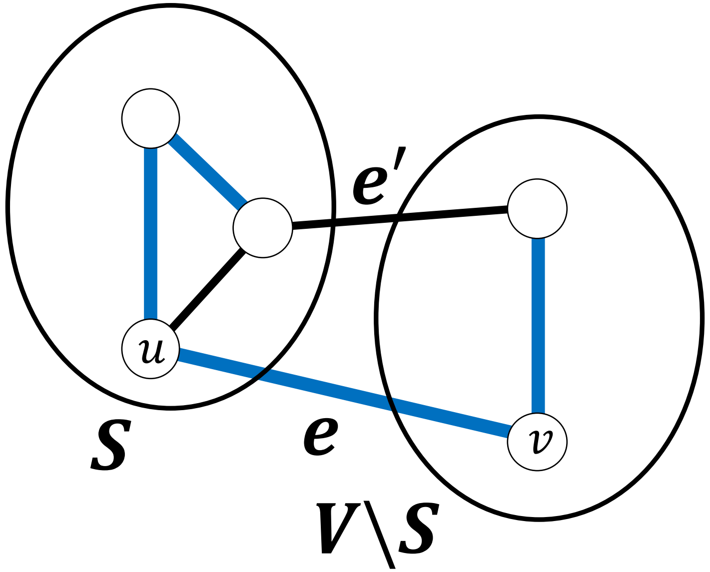

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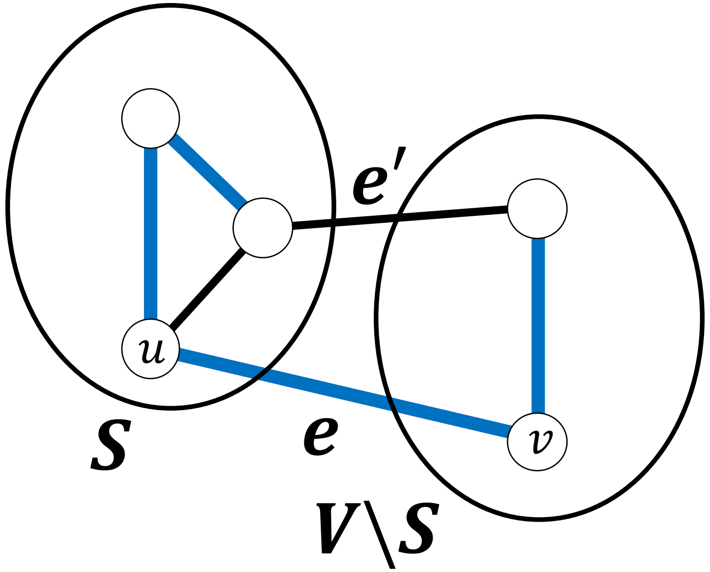
⇒ The MST must include e.



Kruskal's MST Algorithm

Algorithm: Add the edge with smallest weight, that does not create a cycle.

Proof of optimality: Let $G = (V, E)$, and $T \subseteq E$ be the set of edges resulting from Kruskal's algorithm.

How do we use the Cut Property to show that Kruskal's is optimal?

Lemma: The cheapest edge between $S \subseteq V$ and $V \setminus S$ is part of the MST.

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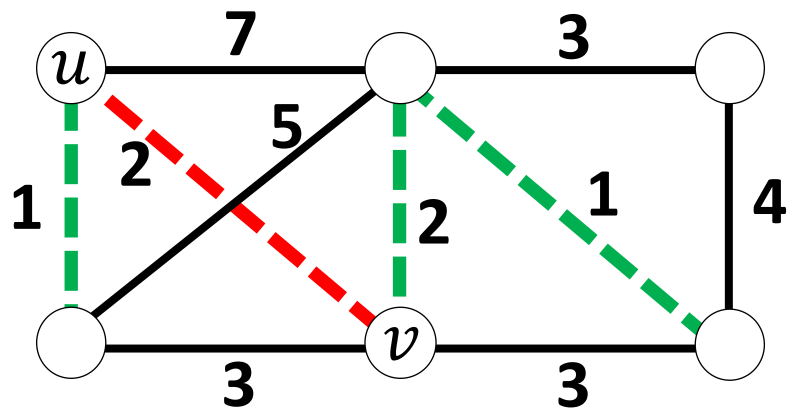

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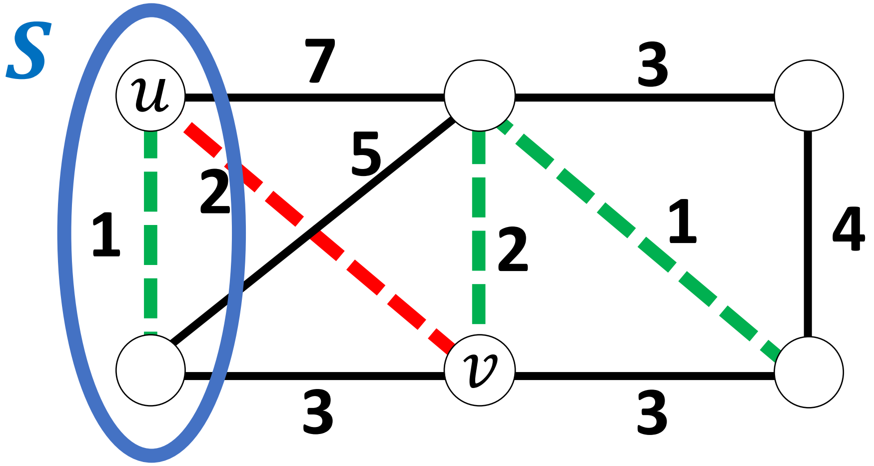

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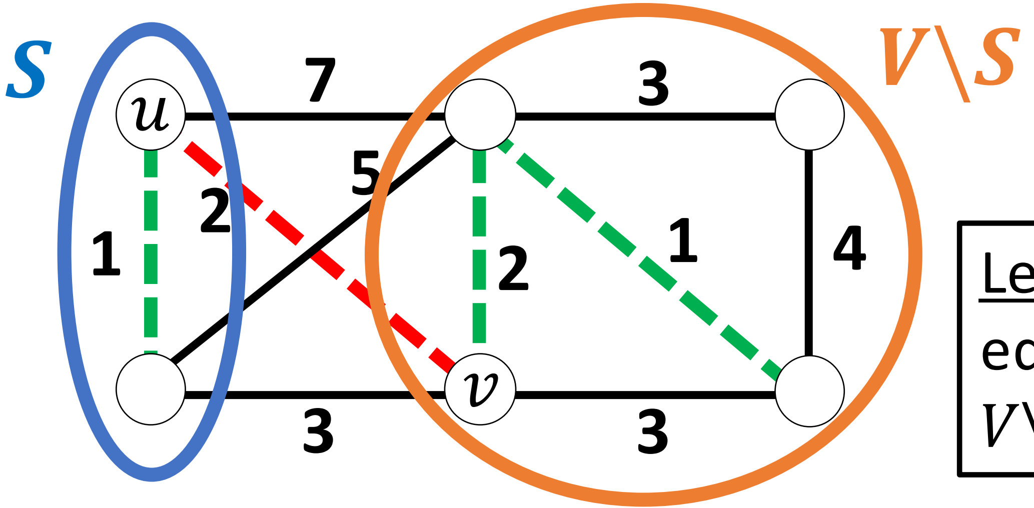

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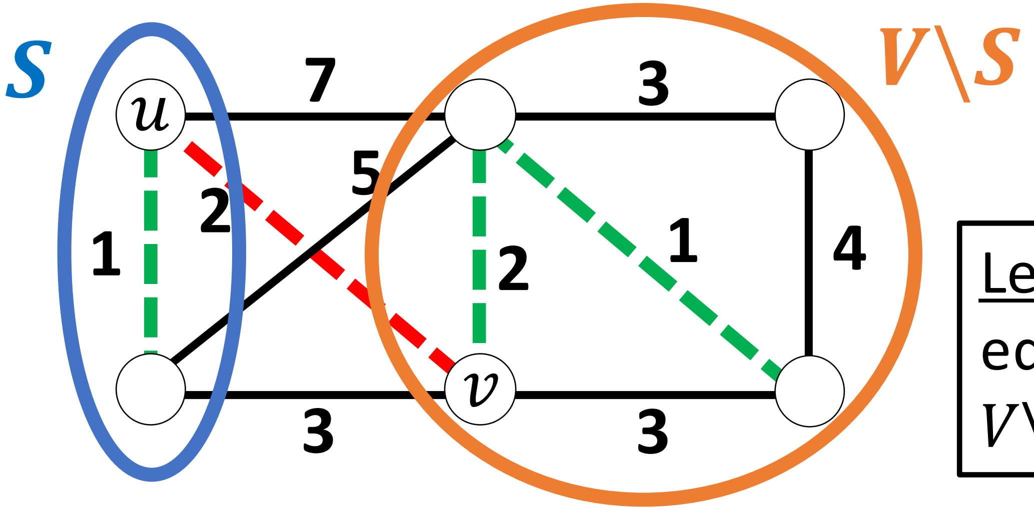

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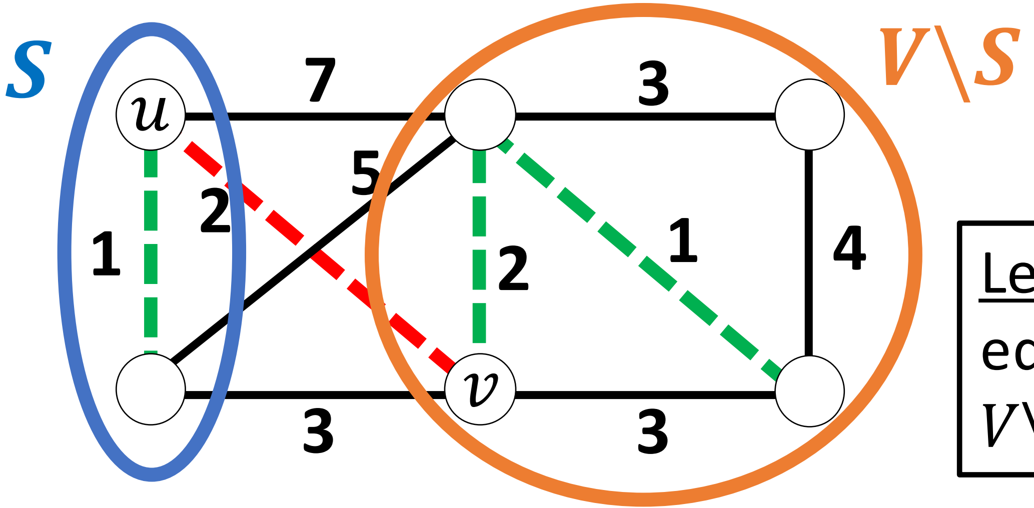

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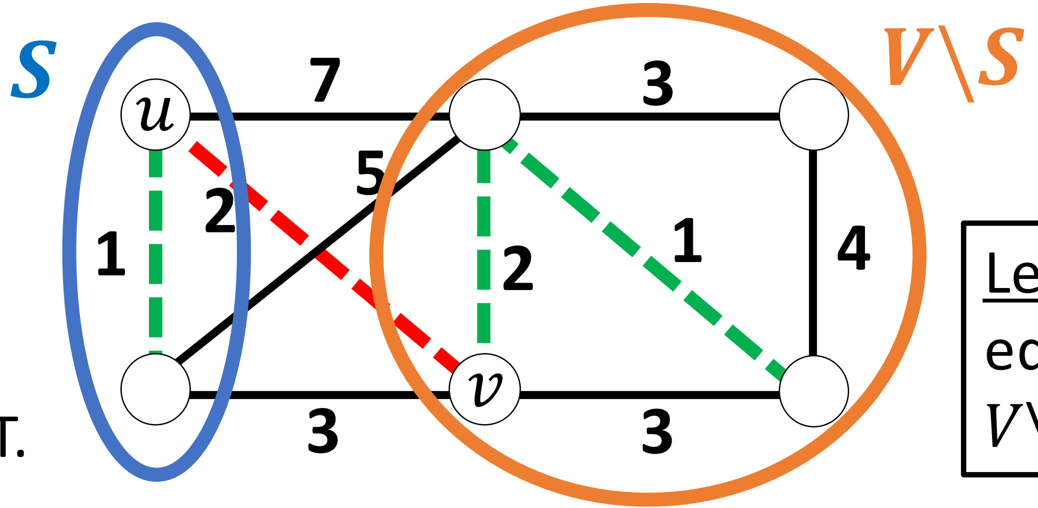
Thus, every edge found by Kruskal's algorithm is part of the MST, and since the edges found form a spanning tree, it is the MST.



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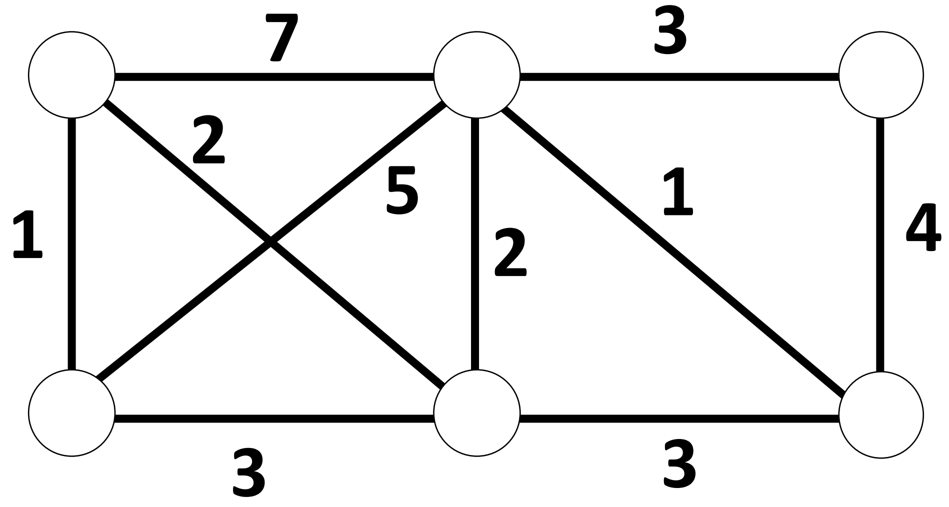
Prim's MST Algorithm

Algorithm: Mark a random node as ***connected***. Find the **edge** with smallest weight between a ***connected*** node and one that is not. Mark both endpoints as ***connected***.



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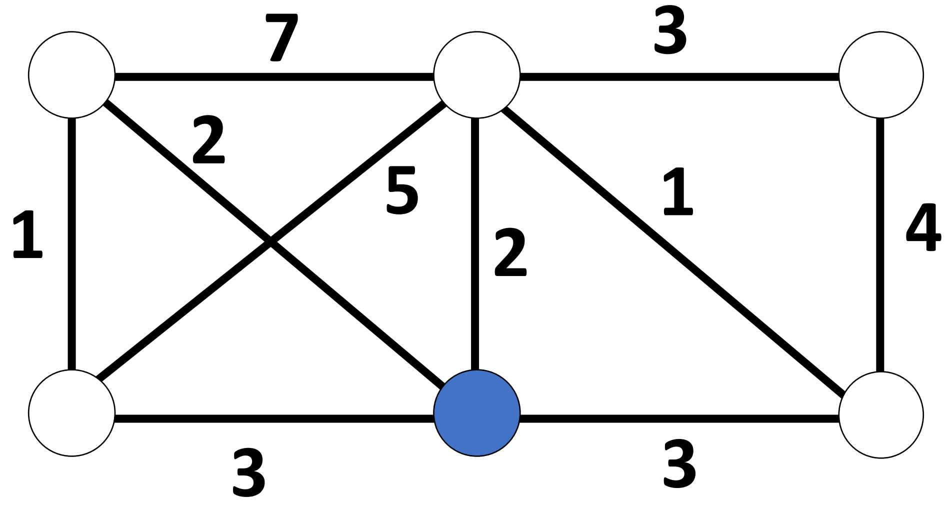

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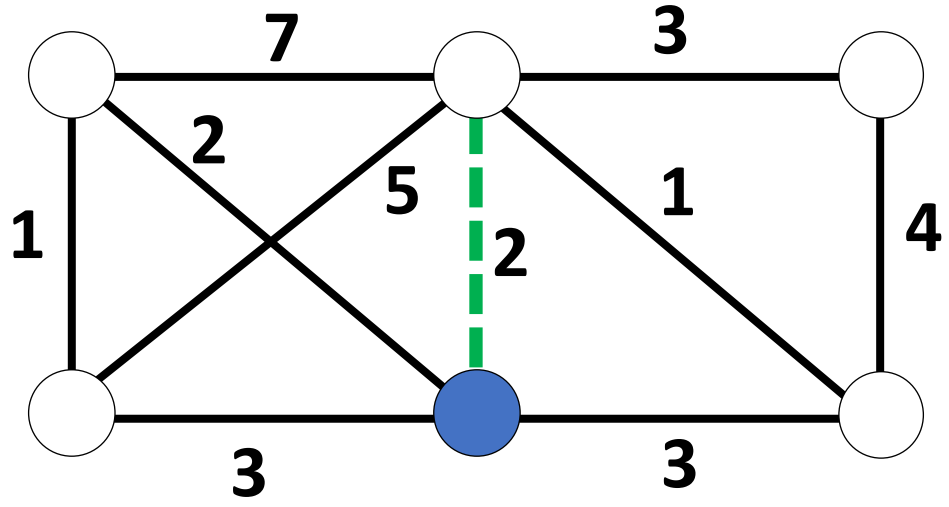

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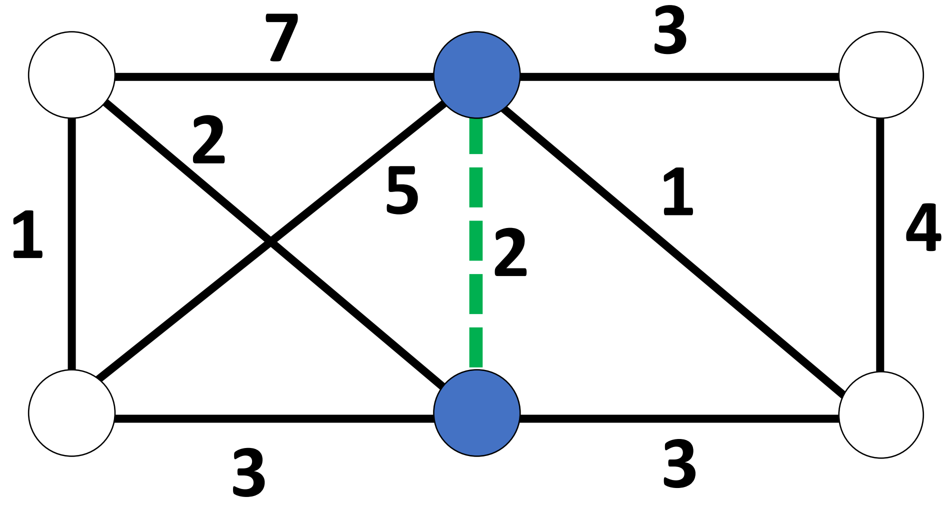

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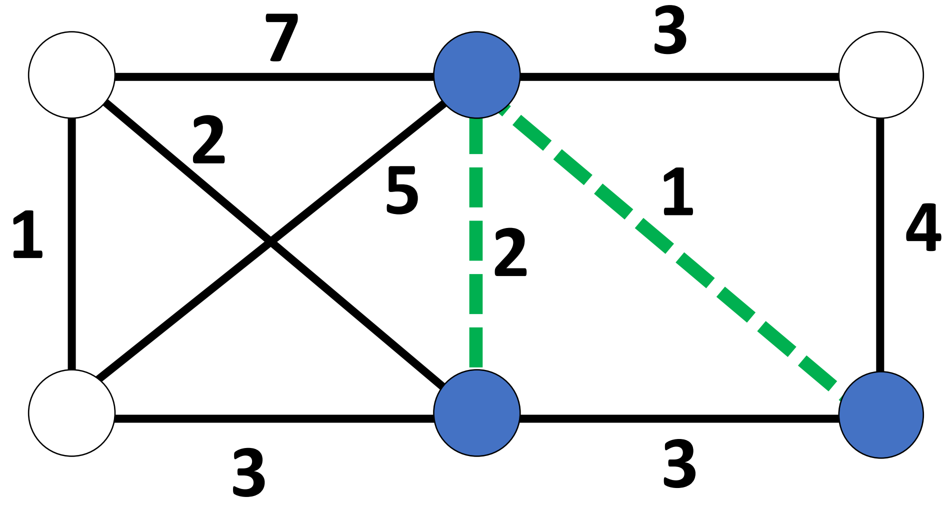

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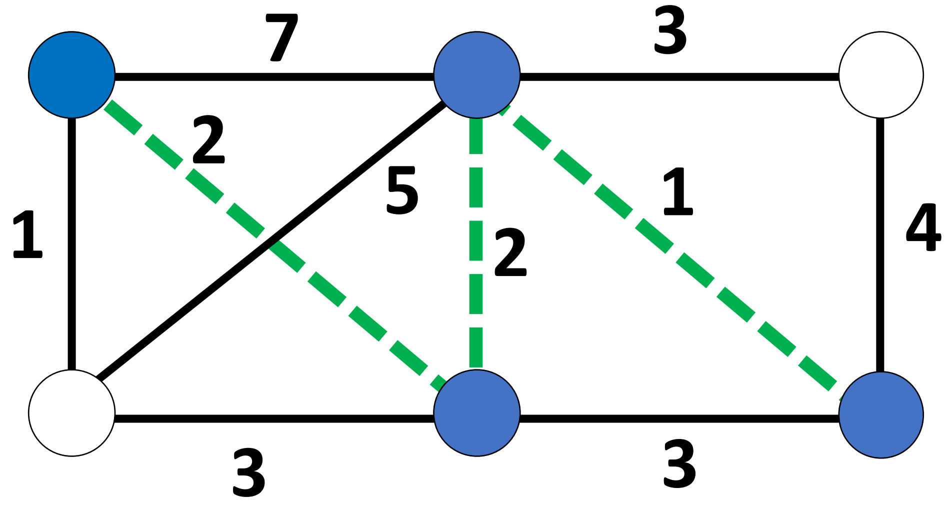

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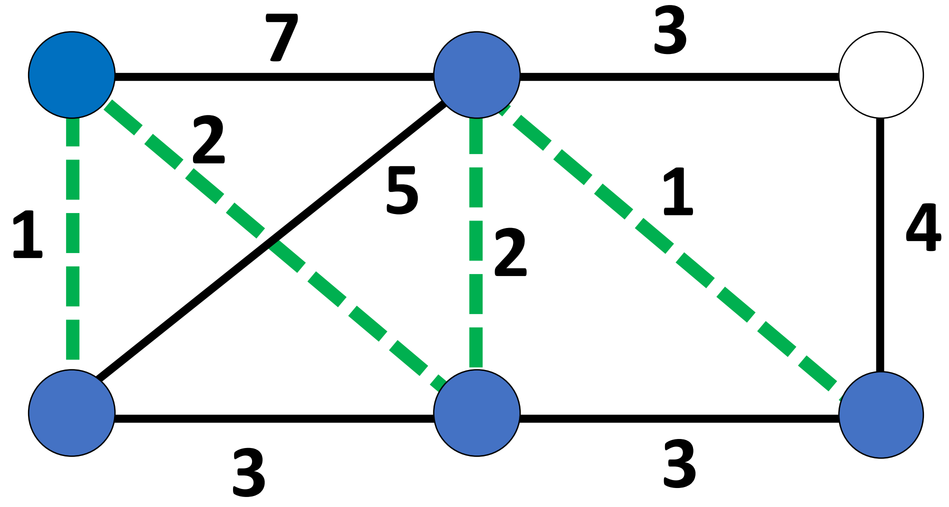

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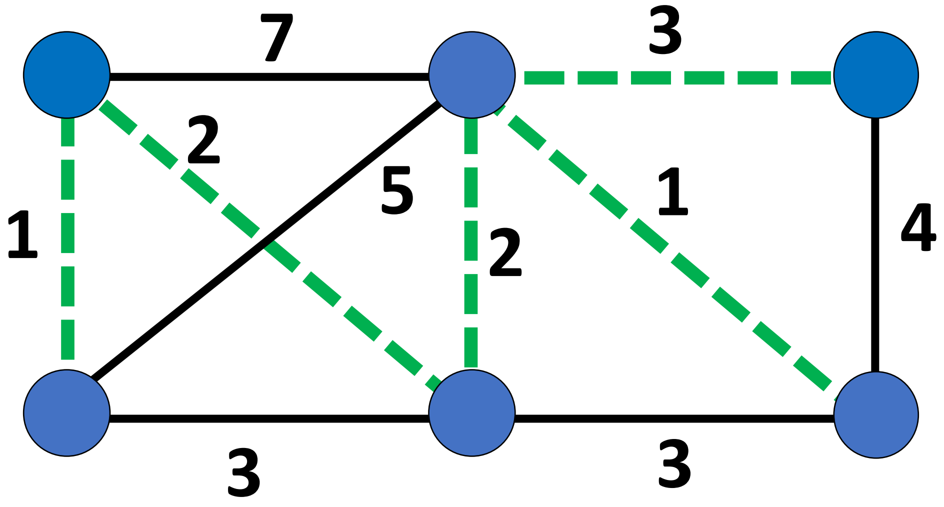

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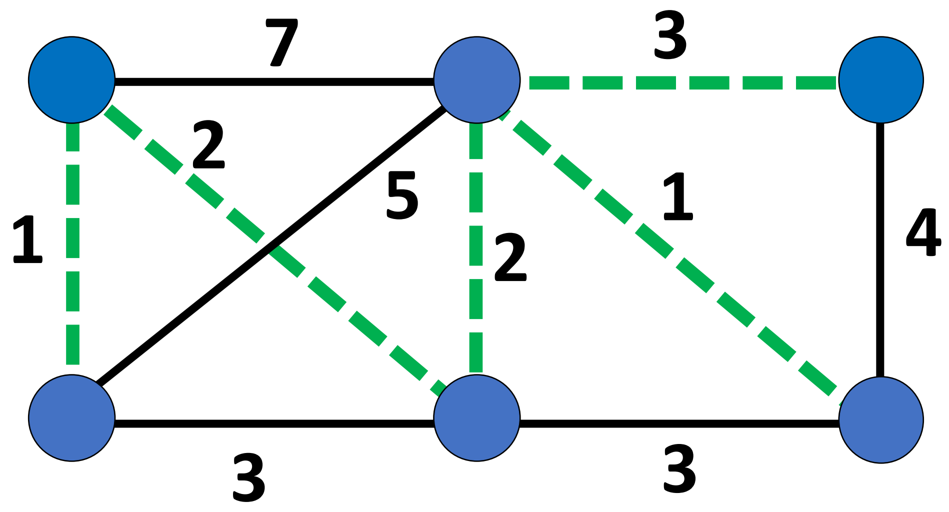

Homework Questions:

1. Is the solution valid? (Does it actually find a spanning tree?)
2. Is the solution optimal?

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