Integer Linear Programming
CSCI 532

Solving ILPs

$x_1, x_2 \in \mathbb{Z}$

Objective: $\min x_1 + x_2$

Subject to: $x_1 \leq 1$

$x_2 \leq 1$

$x_1, x_2 \geq 0$

Can we optimally solve this ILP?

Solving ILPs

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Solving ILPs

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Objective: $\min -x_1 - x_2$

Subject to:
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$$x_2 \leq 1$$
$$x_1, x_2 \geq 0$$

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Solving ILPs

$x_1, x_2 \in \mathbb{Z}$

Objective: $\min -\frac{\pi}{\sqrt{2}} x_1 + \frac{\log 2}{e} x_2 + \frac{32}{17}$

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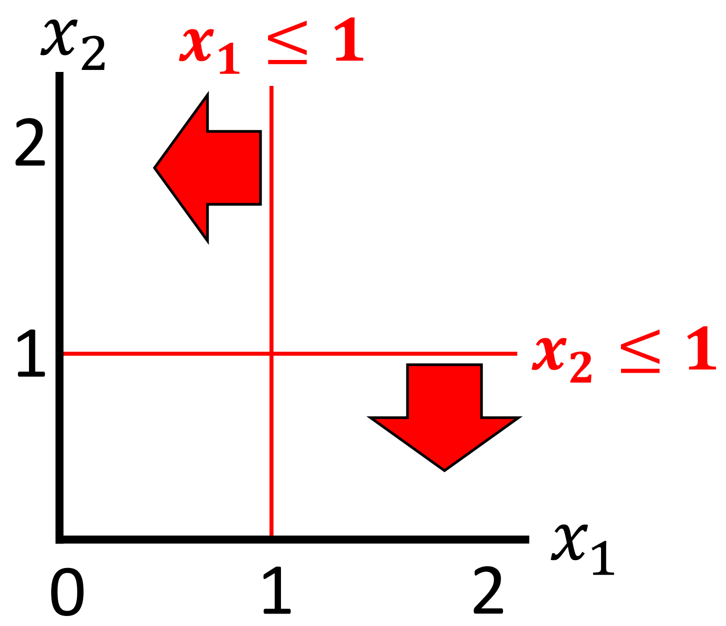

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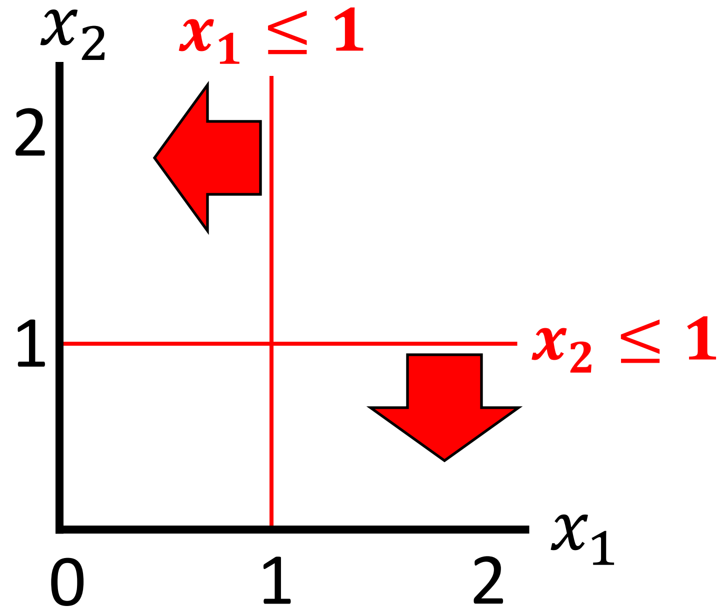

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Why?

If all feasible region vertices are integer-valued, an optimal solution is integer-valued, regardless of the objective function.

Vertex Cover

$x_i \in \{0,1\}$ = Indicates if vertex i is selected.

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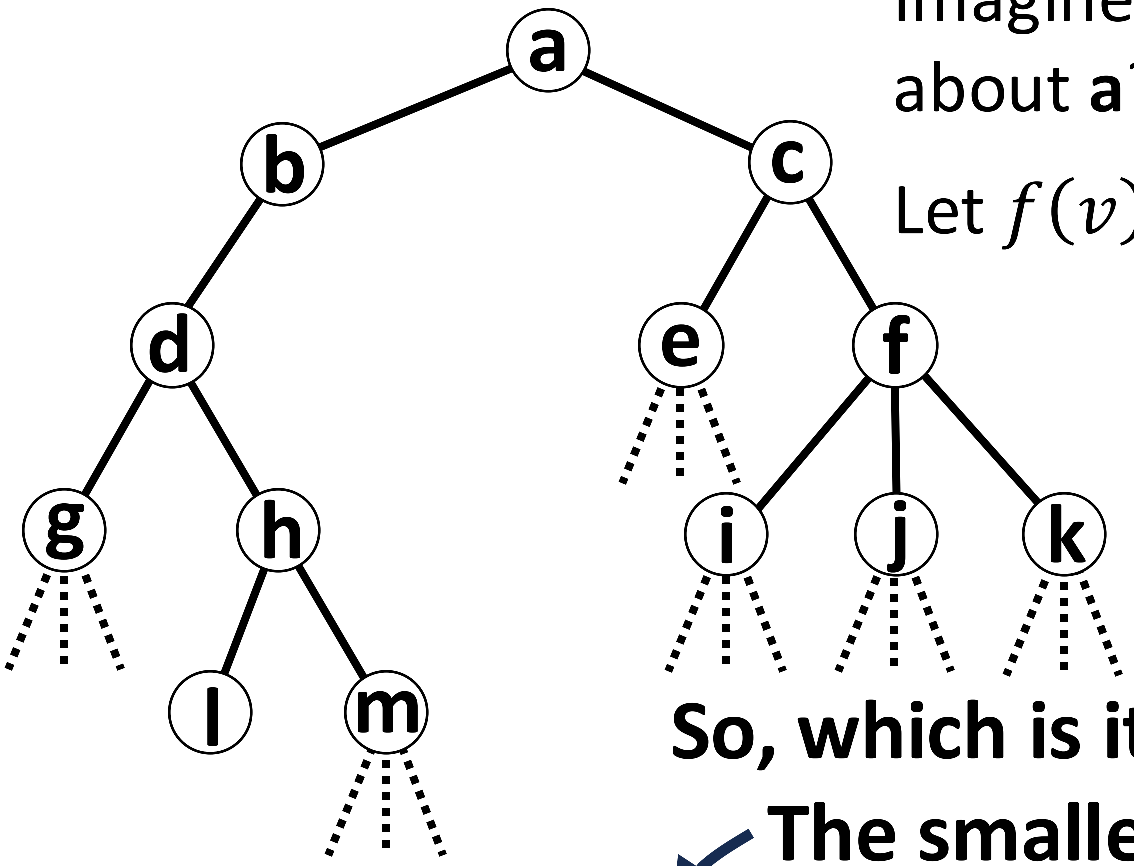
General Graphs: NP-Hard

Vertex Cover in Trees



Imagine the minimum vertex cover. What can we say about **a**? **a** is in a minimum vertex cover, or it's not.

Let $f(v)$ = Size of minimum vertex cover rooted at v.

If a **<u>is</u>** in a minimum VC

$$f(a) = 1 + f(b) + f(c)$$

If a **<u>is not</u>** in a minimum VC

$$f(a) = 2 + f(d) + f(e) + f(f)$$

So, which is it?

The smallest!

$$f(v) = \min \left\{ 1 + \sum_{u \in \text{child}(v)} f(u),\ |\text{child}(v)| + \sum_{w \in \text{grandchild}(v)} f(w) \right\}$$

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Goal: Find some property of special ILPs that give them integer-vertexed feasible regions.

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Theorem (Hoffman-Kruskal, 1956):

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Objective:	max $c^T x$
Subject to:	$A x \leq b$
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If A is totally unimodular, the ILP can be solved in polynomial time!

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$$\begin{pmatrix} 4 & 5 & 7 & 3 & 0 & 1 \\ 1 & 2 & 8 & 4 & 6 & 5 \\ 9 & 9 & 1 & 5 & 7 & 3 \\ 9 & 5 & 6 & 6 & 8 & 2 \\ 1 & 1 & 0 & 1 & 2 & 7 \end{pmatrix} \Rightarrow \begin{bmatrix} 2 & 8 \\ 9 & 1 \end{bmatrix} \begin{bmatrix} 6 & 8 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 5 & 3 \end{bmatrix}$$

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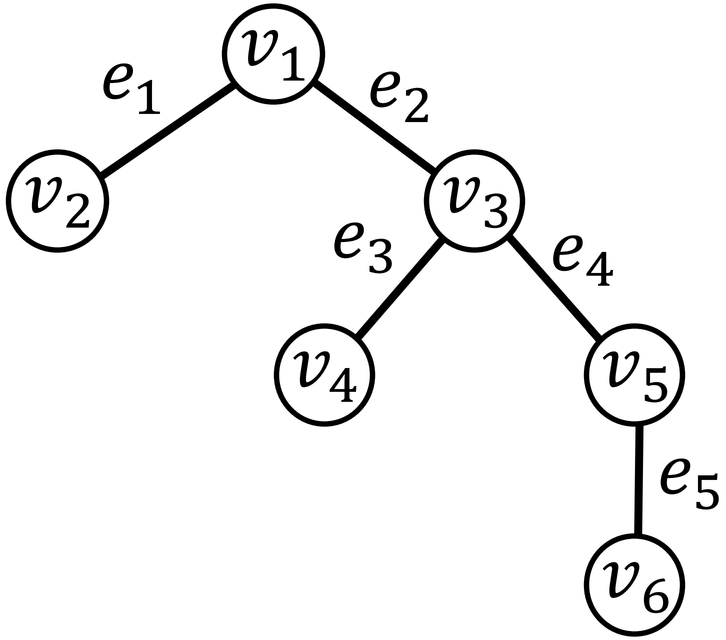

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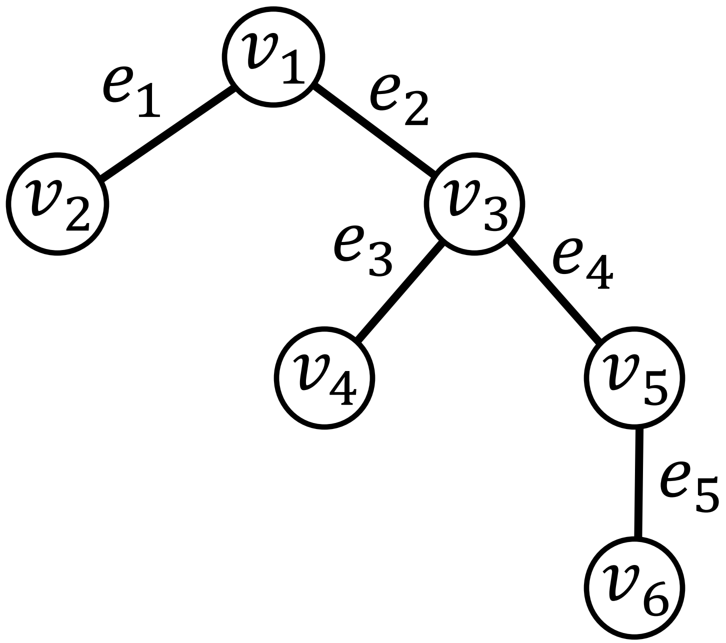

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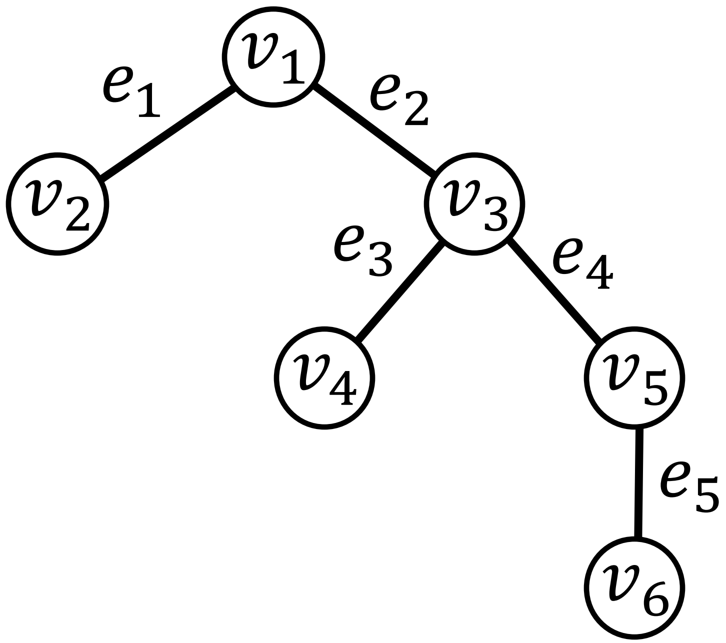

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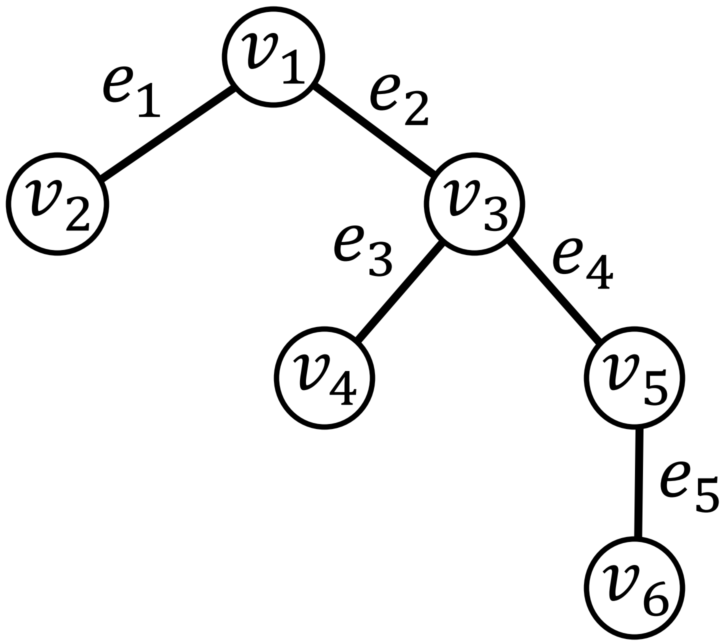

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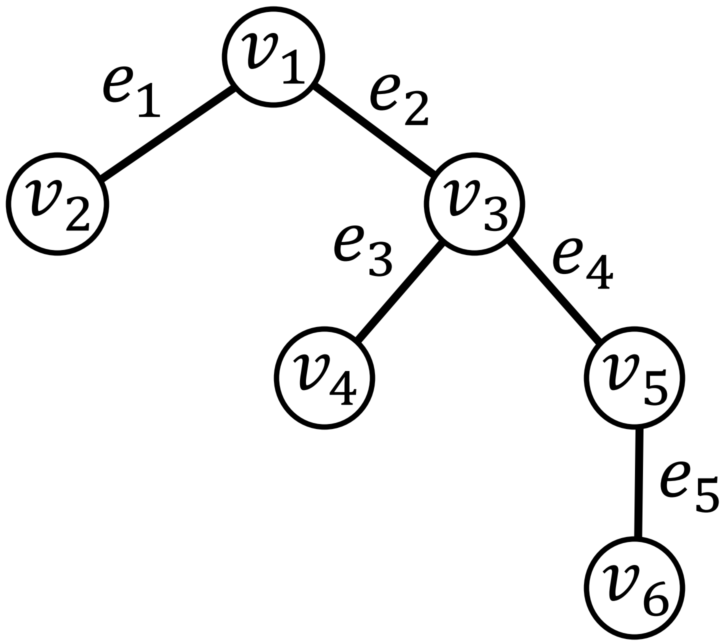

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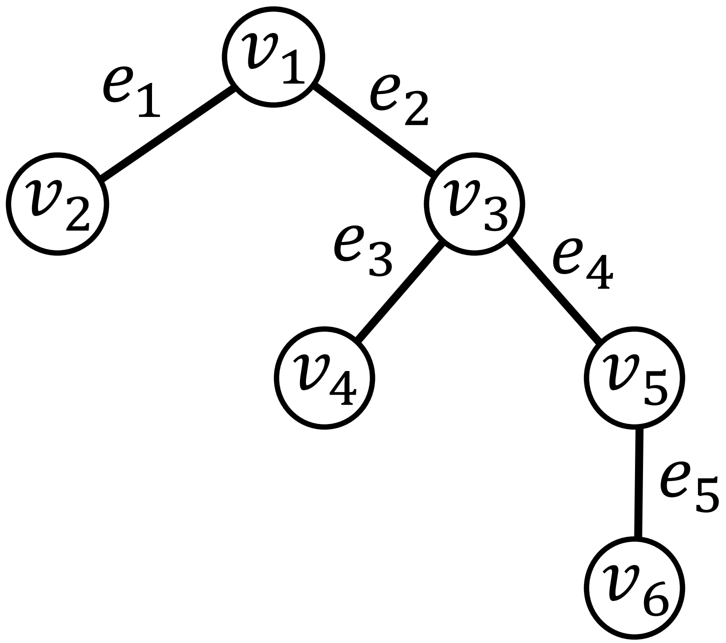

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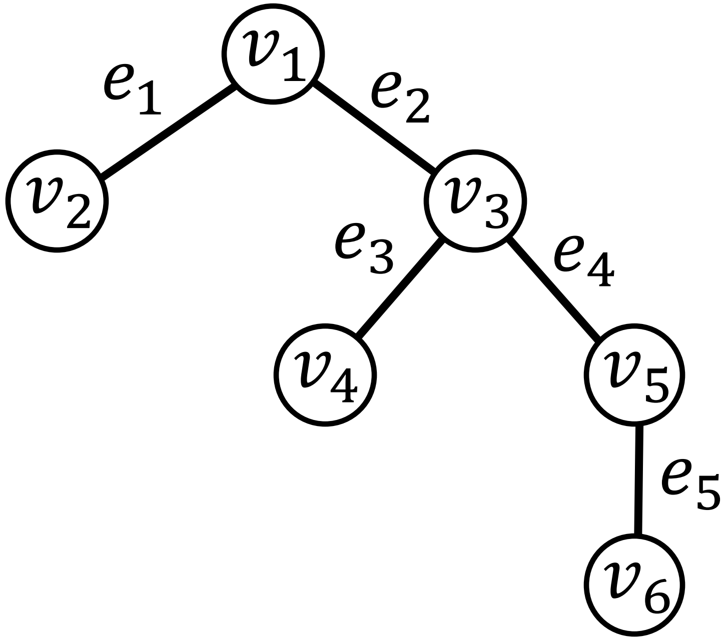
Label alternating generations of vertices into two sets B and G.



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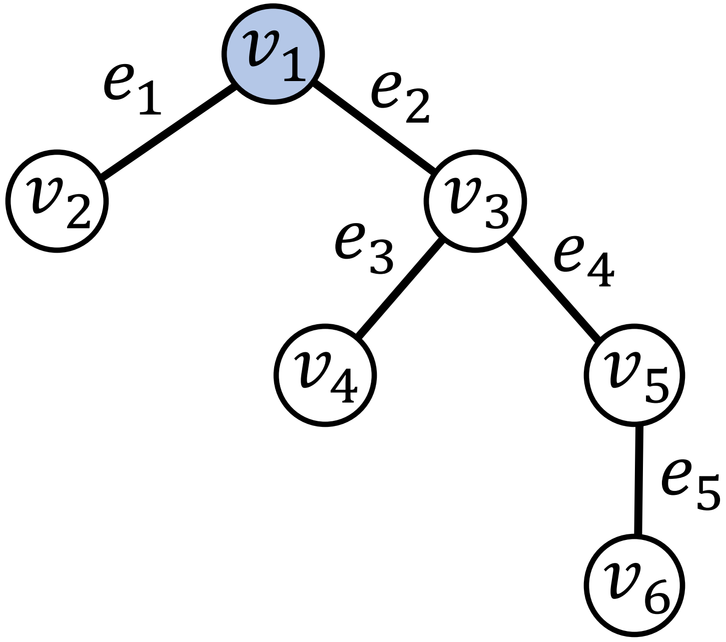

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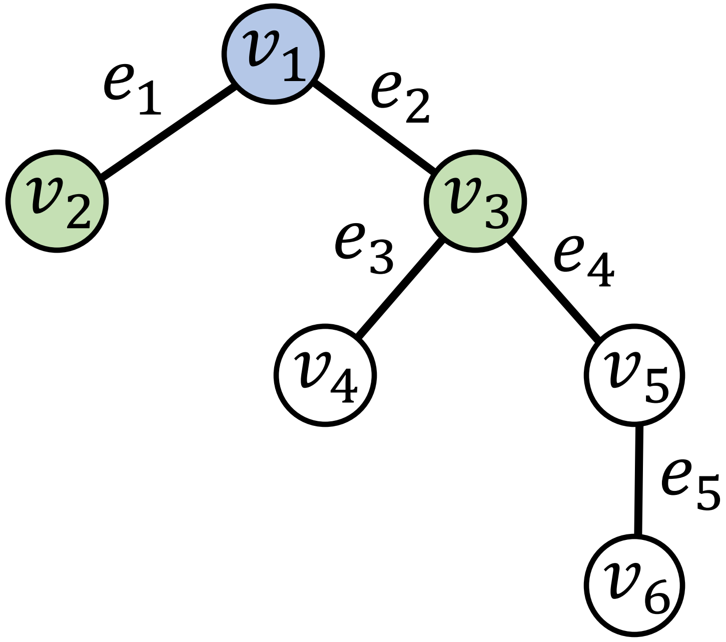

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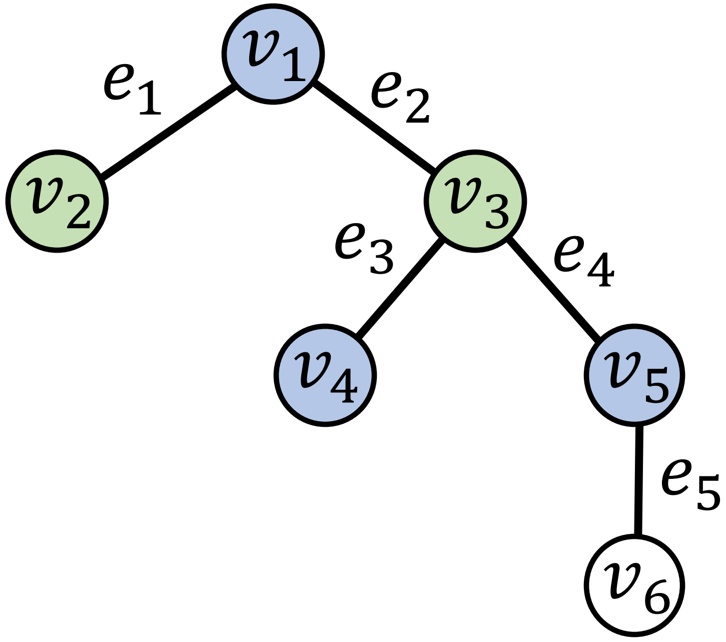

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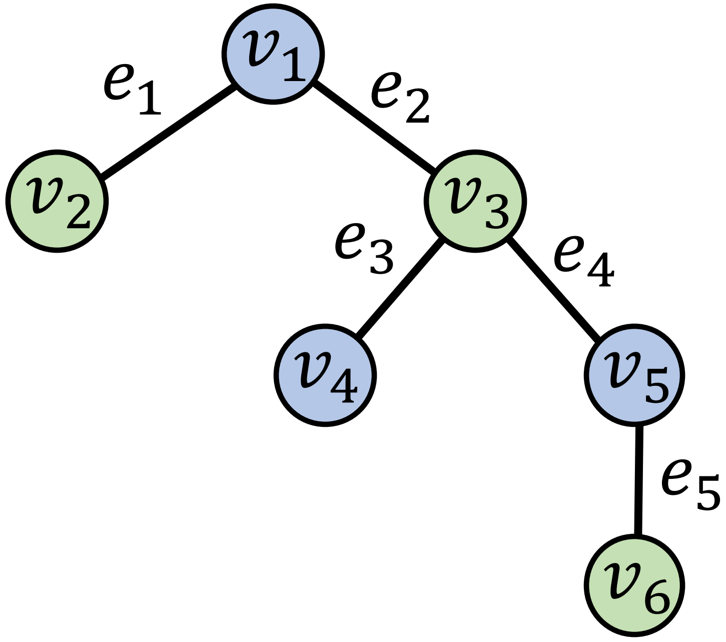

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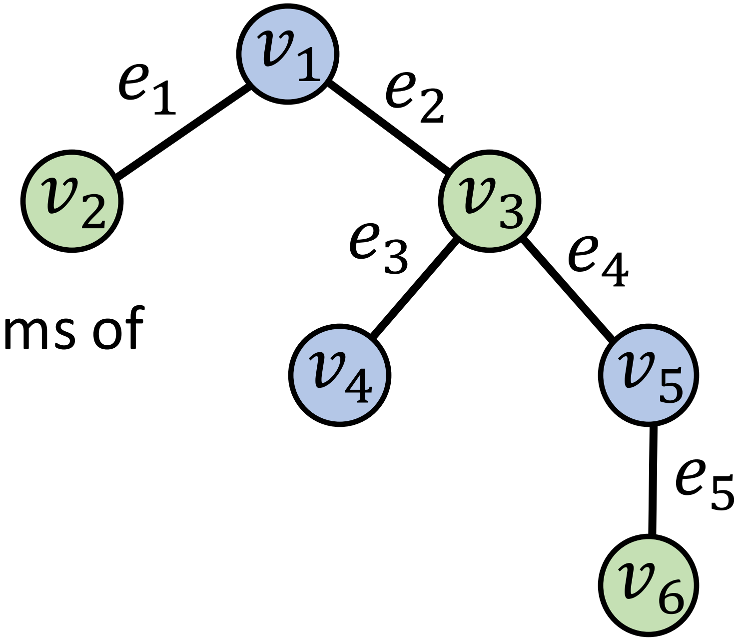
Label alternating generations of vertices into two sets B and G.



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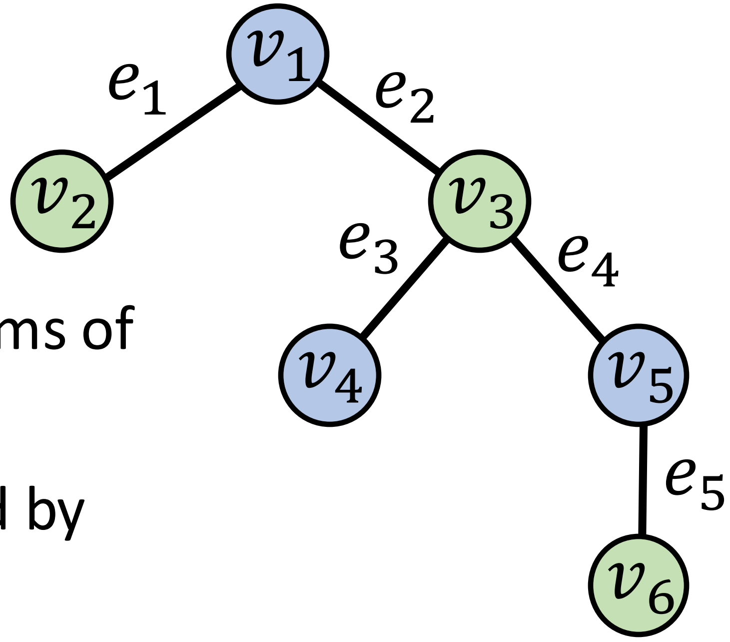

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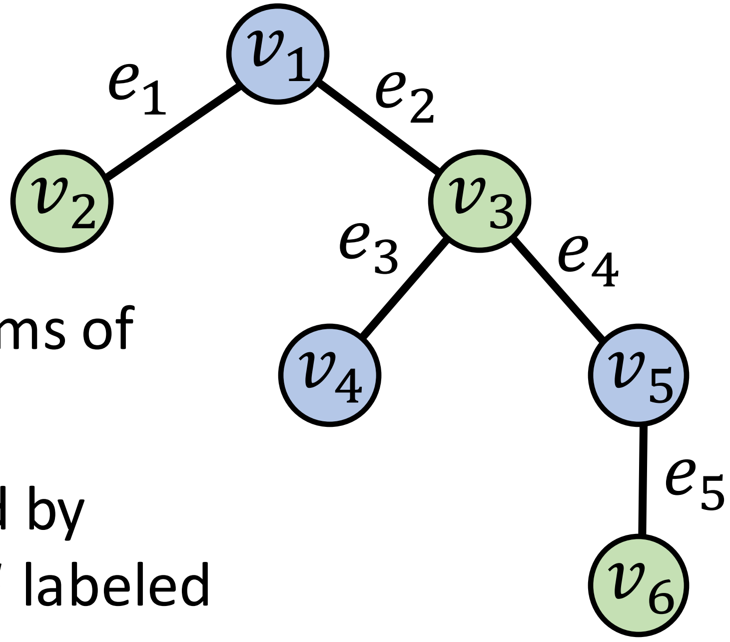

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Therefore, A^{T} and A are totally unimodular.

Vertex Cover

$x_i \in \{0,1\}$ = Indicates if vertex i is selected.

Objective: $\min \sum_i x_i$

Subject to: $x_i + x_j \geq 1$, for each edge $e = (i, j)$

Objective: max $c^T x$

Subject to: $A x \leq b$

$x \geq 0$

Theorem (Hoffman-Kruskal, 1956):

$A \in \mathbb{R}^{m \times n}$ is totally unimodular $\Leftrightarrow$ The feasible region: $\{x \in \mathbb{R}^n | Ax \leq b, x \geq 0\}$ has integer vertices $\forall b \in \mathbb{Z}^m$.

∴ The Vertex Cover ILP will be solved in polynomial time when the graph is a tree.

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Does this work for any other special graphs beyond trees?

Vertex Cover

Theorem (Ghouila-Houri, 1962): $A \in \mathbb{R}^{m \times n}$ is totally unimodular $\Leftrightarrow$ For every subset of rows R, there is a partition $R = R_1 \cup R_2$ such that for every column j,

$$\sum_{i \in R_1} A_{ij} - \sum_{i \in R_2} A_{ij} \in \{-1,0,1\}$$

$$A^{\mathsf{T}} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Bipartitie!

Label alternating generations of vertices into two sets B and G. Consider a subset of rows of A^{T}. Partition R into $R_1 = R \cap B$ and $R_2 = R \cap G$.

For any column of A^{T}, which corresponds to an edge e,

$$\sum_{v \in R_1} A_{ve} \in \{0,1\} \text{ and } \sum_{v \in R_2} A_{ve} \in \{0,1\} \Rightarrow \sum_{v \in R_1} A_{ve} - \sum_{v \in R_2} A_{ve} \in \{-1,0,1\}$$

Does this work for any other special graphs beyond trees?

Vertex Cover

$x_i \in \{0,1\}$ = Indicates if vertex i is selected.

Objective: $\min \sum_i x_i$

Subject to: $x_i + x_j \geq 1$, for each edge $e = (i, j)$

But they all use
the same ILP!

General Graphs: NP-Hard
Trees: $O(n)$
Bipartite: Polynomial

Vertex Cover
(General Graph)

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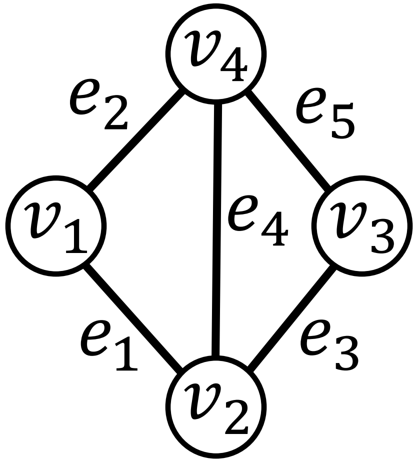

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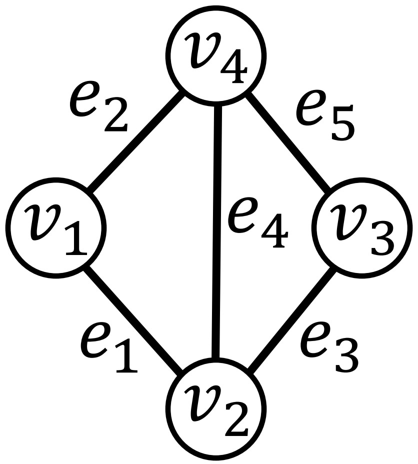

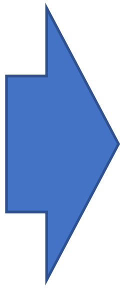

$$A = \begin{array}{c c} & \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{array}{c} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{array} & \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

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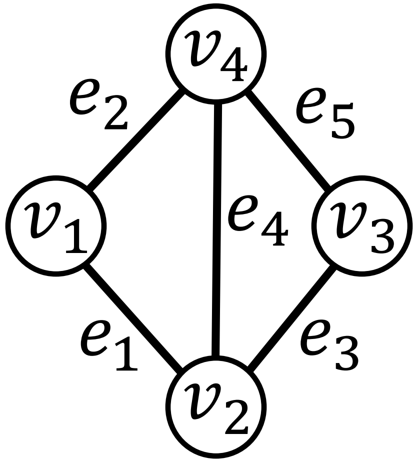

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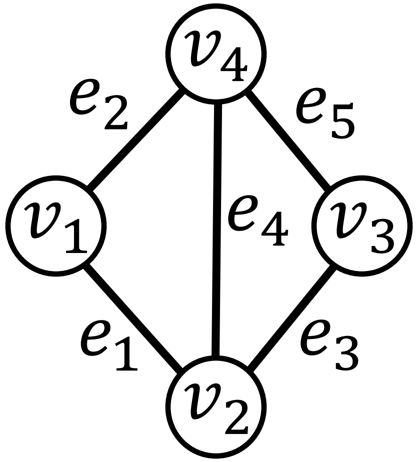

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Where is the contradiction with Ghouila-Houri?

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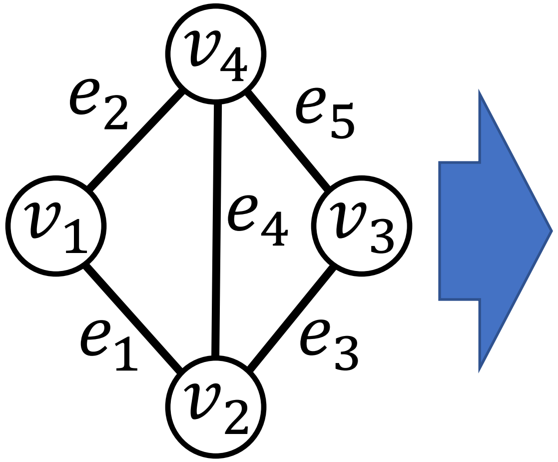

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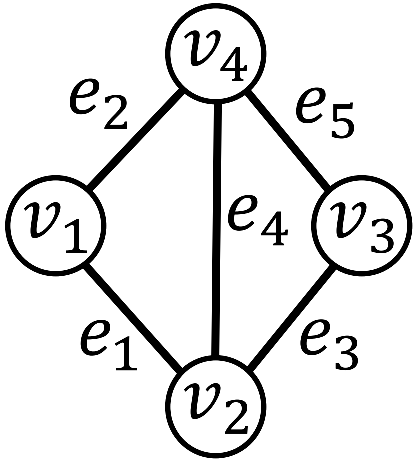

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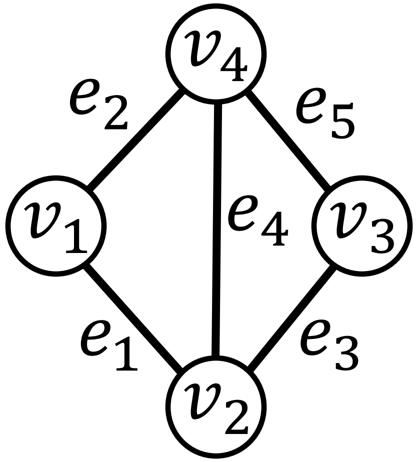

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$R_1 = \{x_1\}$

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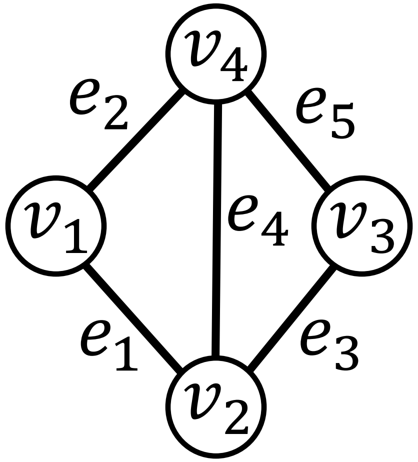

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x_1 and x_2 have to be on different sides of the partition.

$R_1 = \{x_1, x_4\}$
$R_2 = \{x_2\}$

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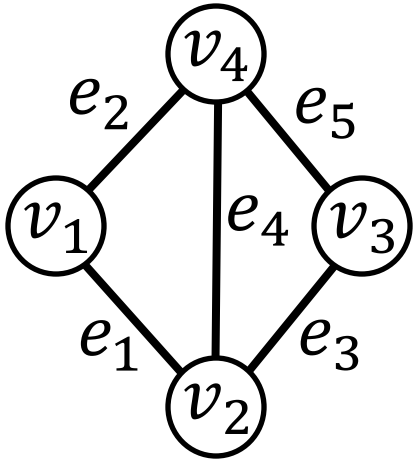

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$$\left.\begin{array}{l} R_1 = \{x_1, x_4\} \\ R_2 = \{x_2\} \end{array}\right\}$$

Column e_2 sums to 2.

$$R_1 = \{x_1\}$$
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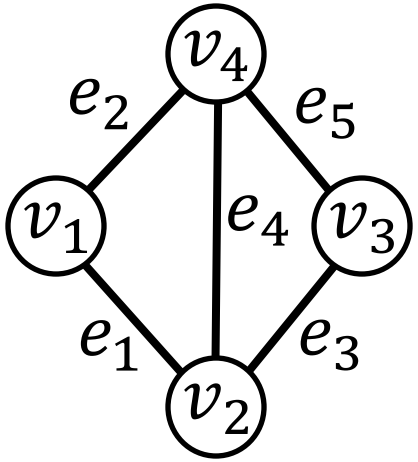

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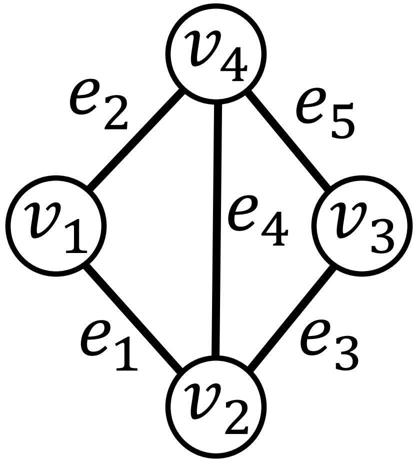

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No such partition exists, so A^{T} and A are not totally unimodular.

Final Notes

Theorem (Hoffman-Kruskal, 1956): $A \in \mathbb{R}^{m \times n}$ is totally unimodular $\Leftrightarrow$ The feasible region: $\{x \in \mathbb{R}^n | Ax \leq b, x \geq 0\}$ has integer vertices $\forall b \in \mathbb{Z}^m$.

Total unimodularity is not the only route to an integer-vertexed feasible region.

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$x_1, x_2 \in \mathbb{Z}$

Objective: $\min x_1 + x_2$

Subject to:
$$\begin{aligned} x_1 + x_2 &\leq 1 \\ x_1 - x_2 &\leq 1 \\ x_1, x_2 &\geq 0 \end{aligned}$$

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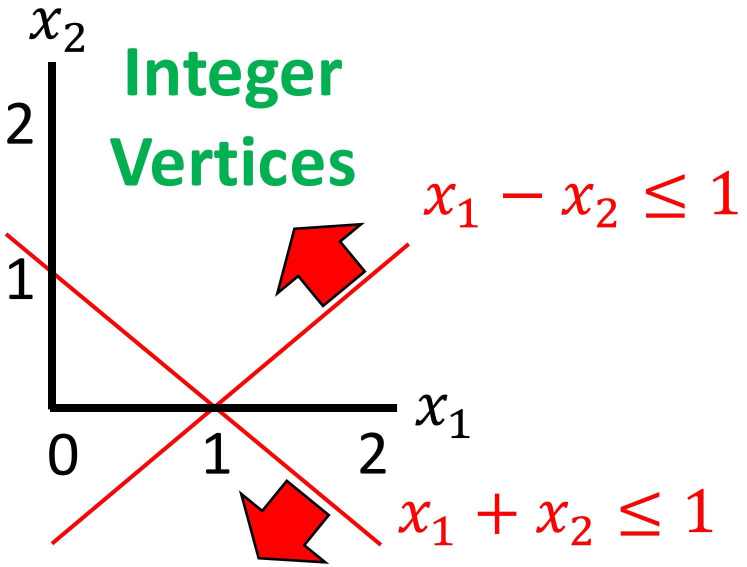

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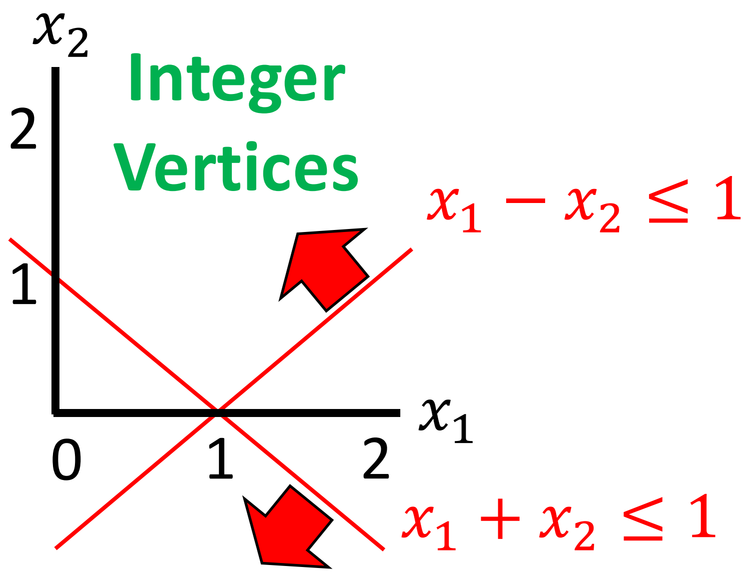

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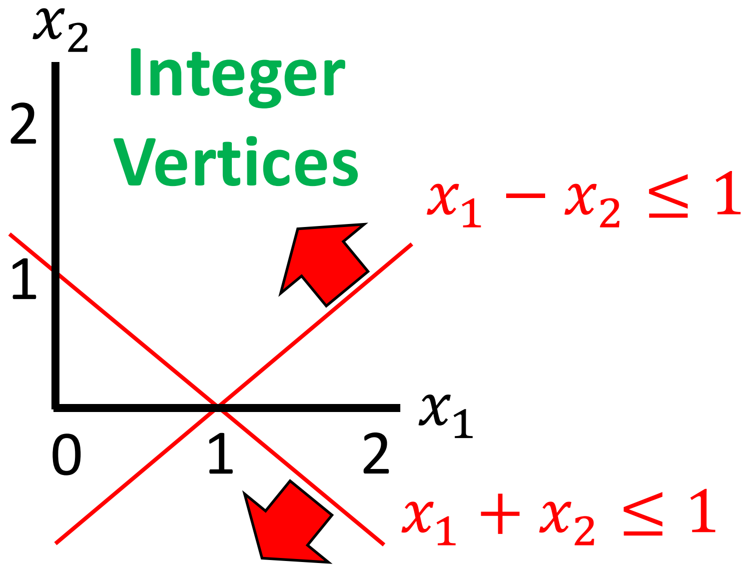

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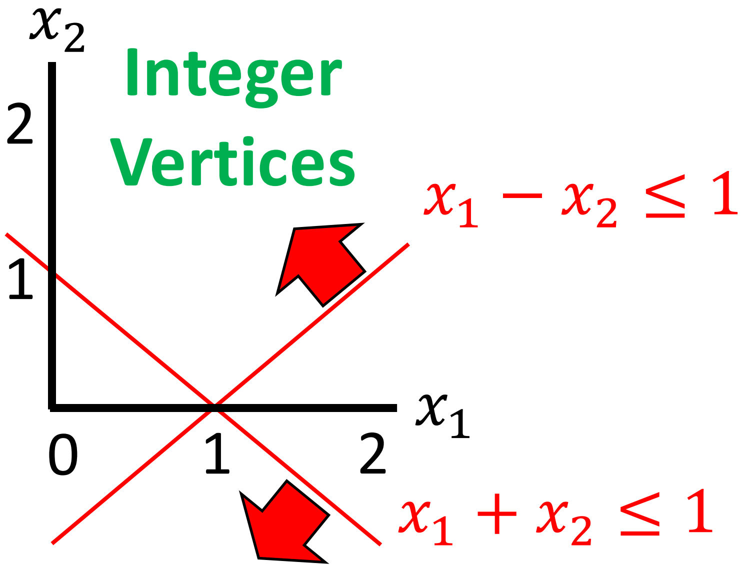

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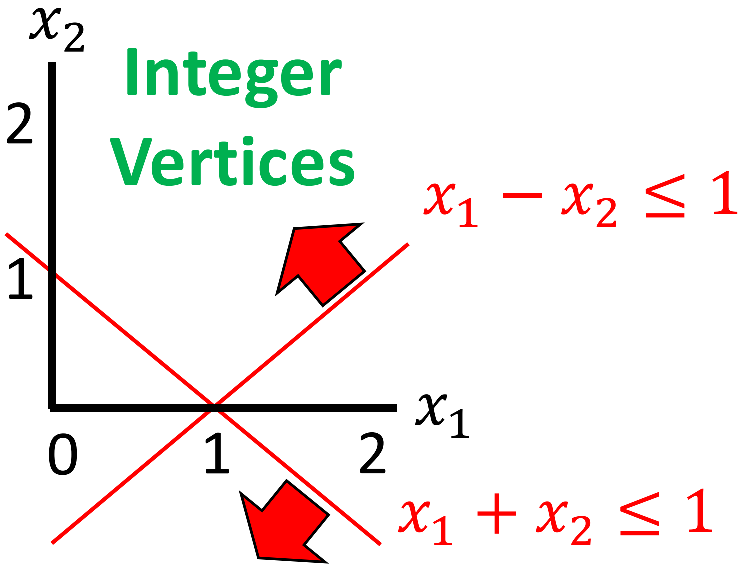
Final Notes

Theorem (Hoffman-Kruskal, 1956): $A \in \mathbb{R}^{m\times n}$ is totally unimodular $\Leftrightarrow$ The feasible region: $\{x \in \mathbb{R}^n | Ax \le b, x \ge 0\}$ has integer vertices $\forall b \in \mathbb{Z}^m$.



$x_1, x_2 \in \mathbb{Z}$

Objective: $\min x_1 + x_2$

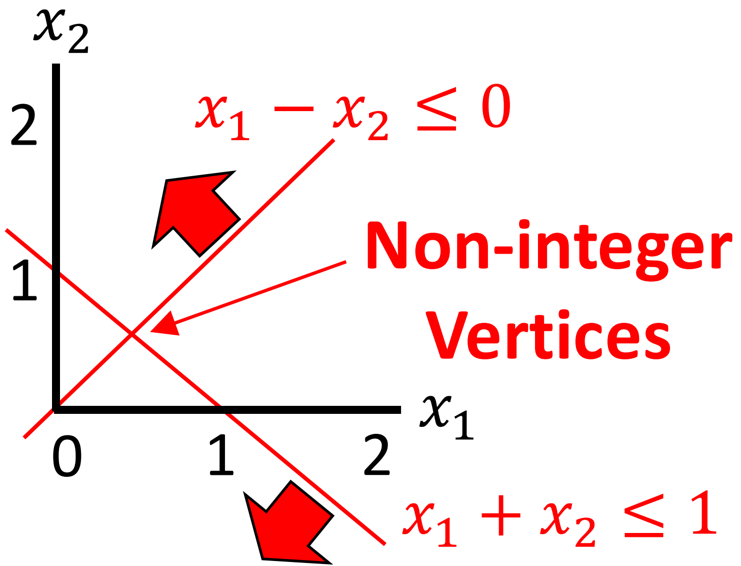
Subject to: $x_1 + x_2 \le 1$

$x_1 - x_2 \le 1$

$x_1, x_2 \ge 0$

$$A = \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{array} \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \end{array}$$

Non-totally Unimodular



$x_1, x_2 \in \mathbb{Z}$

Objective: $\min x_1 + x_2$

Subject to: $x_1 + x_2 \le 1$

$x_1 - x_2 \le 0$

$x_1, x_2 \ge 0$

$$A = \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{array} \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \end{array}$$