Integer Linear Programming
CSCI 532

x_i is 0 or 1, not 0.62

$x_i \in \{0,1\}$ = Indicates if vertex i is selected.

Objective: $\min \sum_i x_i$

Subject to: $x_i + x_j \geq 1$, for each edge $e = (i, j)$

What is this for?

Vertex Cover (VC)

Vertex Cover: Given graph $G = (V, E)$, find the smallest $V' \subseteq V$ such that each edge in E contains an end point in V'?

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What have we accomplished?

Solving ILPs is NP-Hard.

Questions?

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In general

Vertex Cover (VC)

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Subject to: $x_i + x_j \geq 1$, for each edge $e = (i, j)$

<u>Example:</u>

Objective: $\min x_1 + x_2 + x_3 + x_4 + x_5$

Subject to:
$$x_1 + x_2 \geq 1$$
$$x_1 + x_3 \geq 1$$
$$x_2 + x_4 \geq 1$$
$$x_3 + x_4 \geq 1$$
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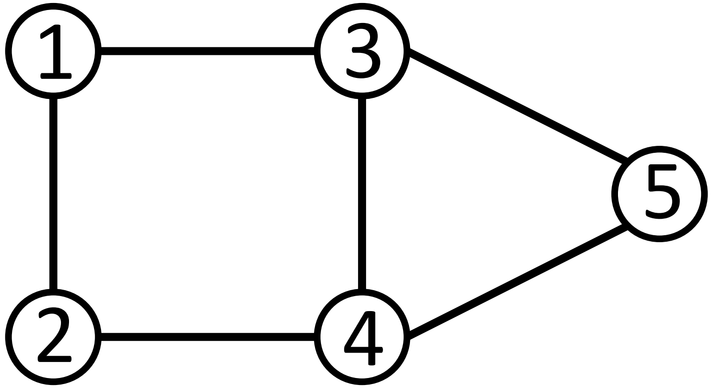

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Subject to: $x_i + x_j \geq 1$, for each edge $e = (i, j)$

Optimal Solution:

$x_1 = 0$
$x_2 = 1$
$x_3 = 1$
$x_4 = 1$
$x_5 = 0$

Objective = 3

<u>Example:</u>

Objective: $\min x_1 + x_2 + x_3 + x_4 + x_5$

Subject to:
$x_1 + x_2 \geq 1$
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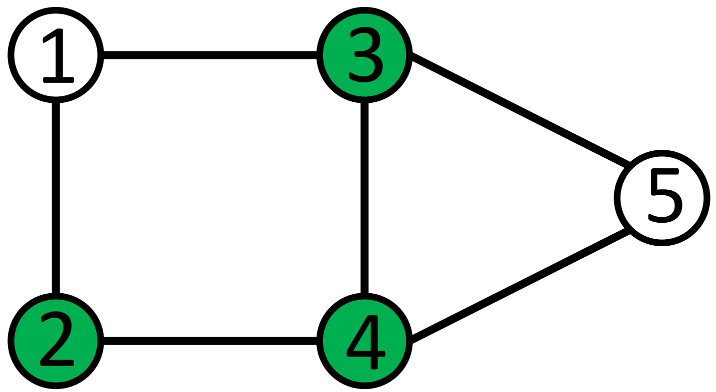

Vertex Cover (VC)

Vertex Cover: Given graph $G = (V, E)$, find the smallest $V' \subseteq V$ such that each edge in E contains an end point in V'?

What happens?

$x_i \in [0,1]$ = Indicates if vertex i is selected.

Objective: $\min \sum_i x_i$

Subject to: $x_i + x_j \geq 1$, for each edge $e = (i, j)$

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$$x_1 + x_2 \geq 1$$
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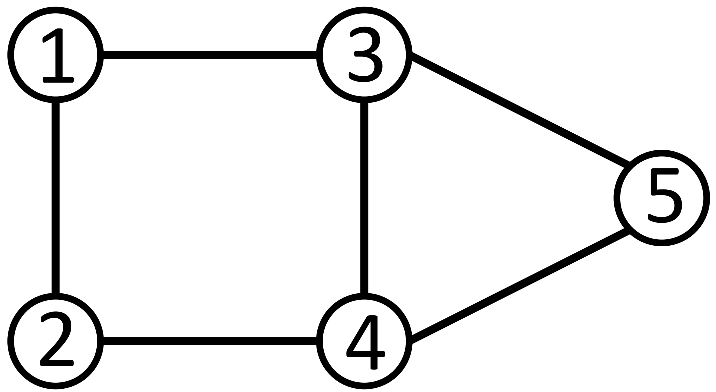

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What happens?

$x_1 = 0.5$

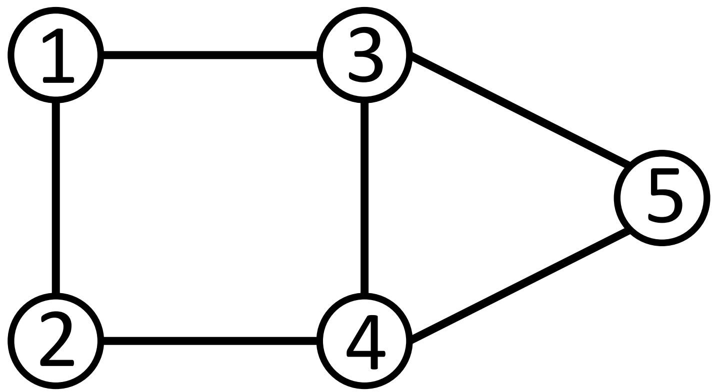
$x_2 = 0.5$

$x_3 = 0.5$

$x_4 = 0.5$

$x_5 = 0.5$

Objective = 2.5



ILP vs LP

$x_i \in \{0,1\}$ = Indicates if vertex i is selected.

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Since the LP has **more options** to reduce the objective value, $OPT_{LP} \leq OPT_{ILP}$ (for a minimization problem).

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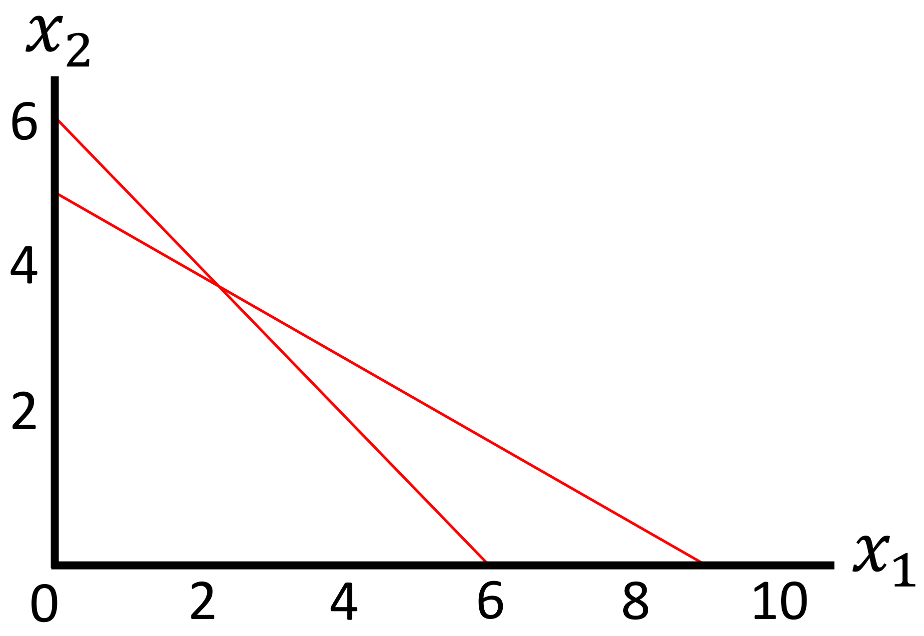
Objective = 2.5

Since the LP has **more options** to reduce the objective value, $OPT_{LP} \leq OPT_{ILP}$ (for a minimization problem). If the minimum objective value comes from an integer solution, a plain LP solver (e.g., Simplex) will find it.

$x_1, x_2 \in \mathbb{R}$

Objective:	$\max 5x_1 + 8x_2$
Subject to:	$x_1 + x_2 \leq 6$
	$5x_1 + 9x_2 \leq 45$
	$x_1, x_2 \geq 0$

Why are ILPs hard to solve?



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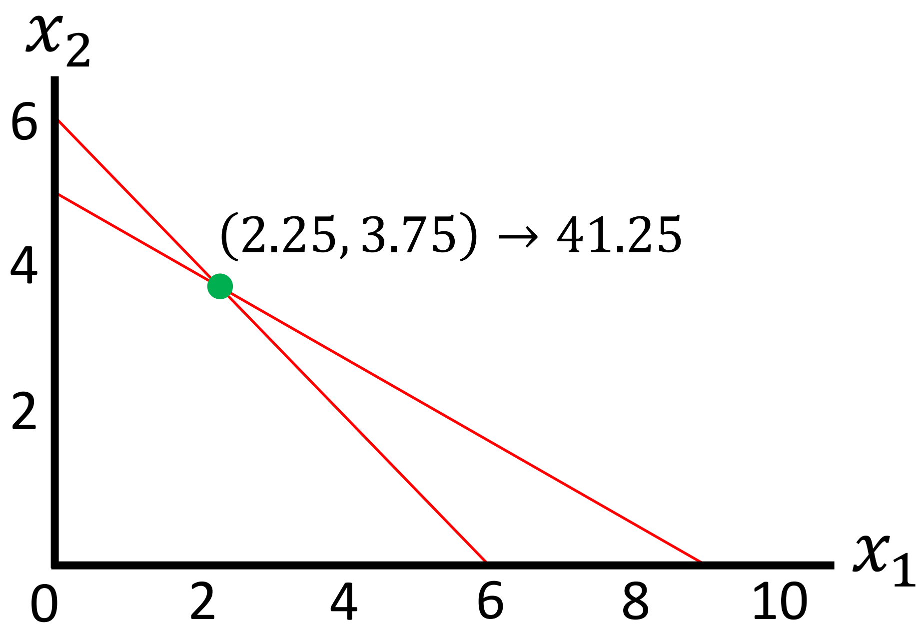

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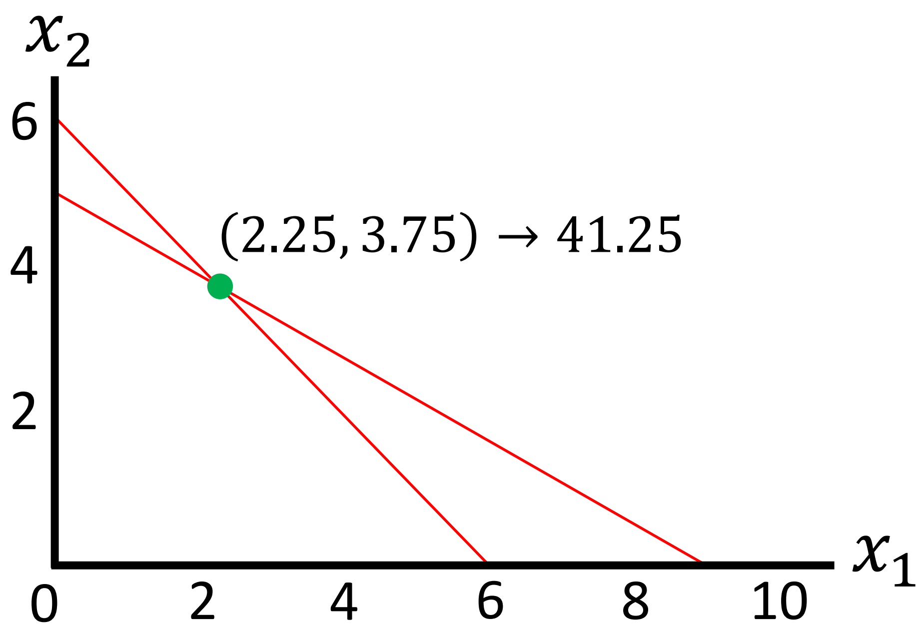

$x_1, x_2 \in \mathbb{N}$

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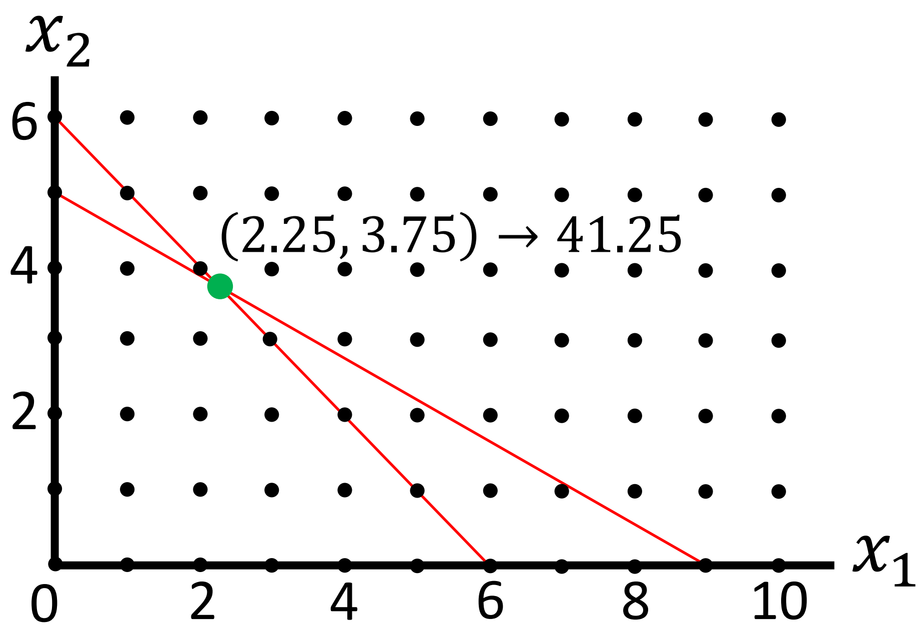

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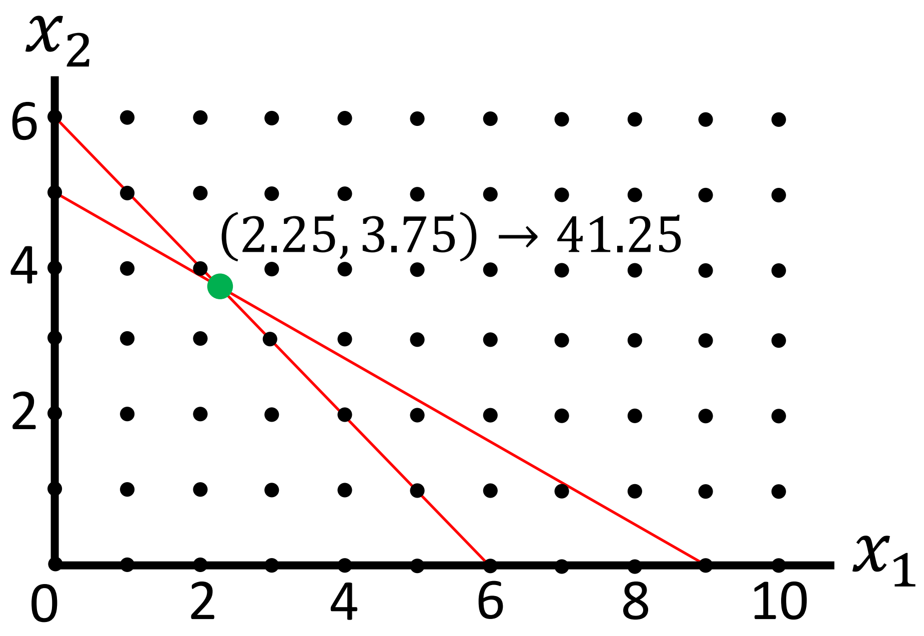

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Optimal continuous solution $\rightarrow$ optimal integer solution?

- Closest integer solution?
- Closest feasible integer solution?
- Closest feasible integer solution on feasible region boundary?

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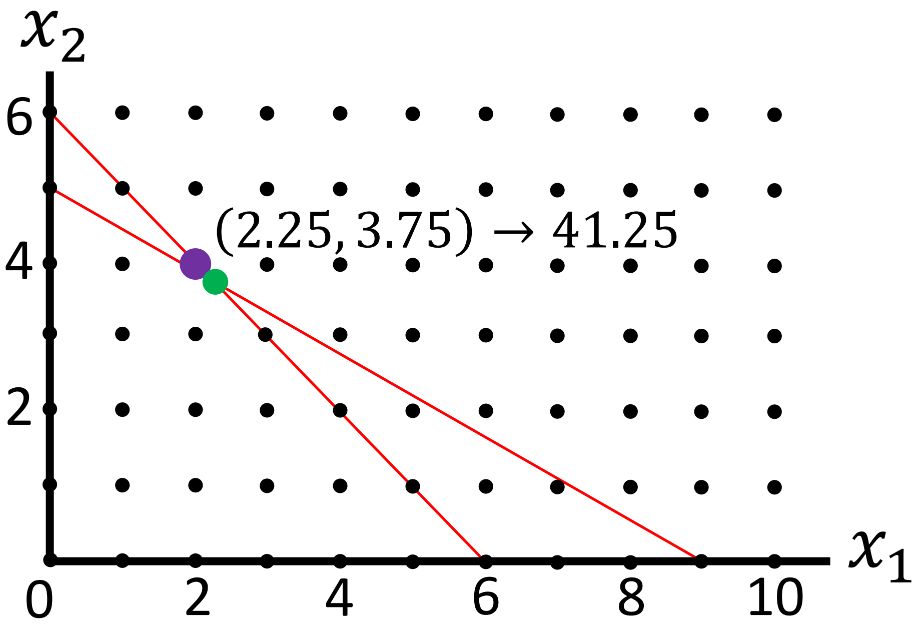

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- Closest integer solution? – Not feasible
- Closest feasible integer solution?
- Closest feasible integer solution on feasible region boundary?

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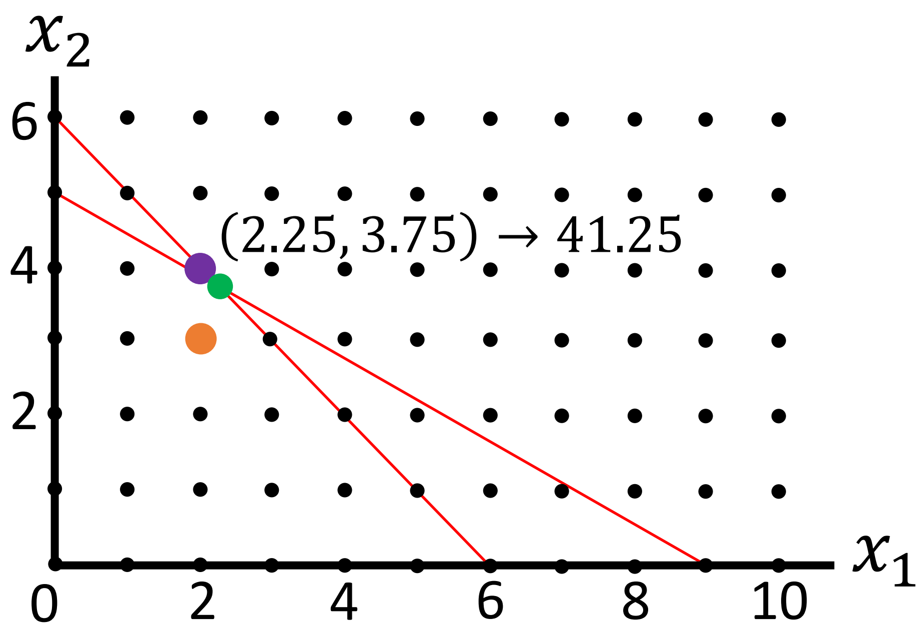

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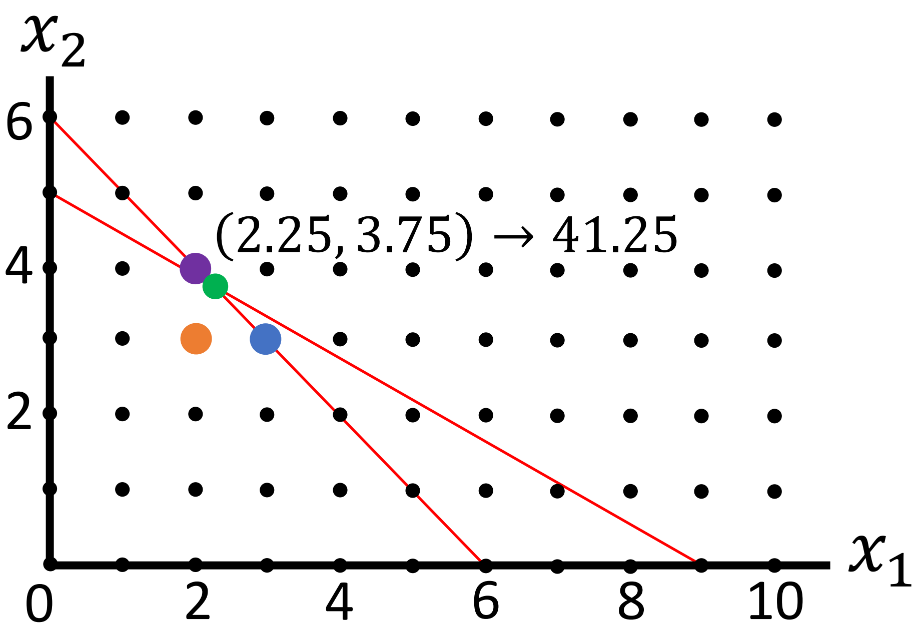
Optimal continuous solution → optimal integer solution?

- Closest integer solution? – Not feasible
- Closest feasible integer solution? – Obj = 34
- Closest feasible integer solution on feasible region boundary?

Why are ILPs hard to solve?



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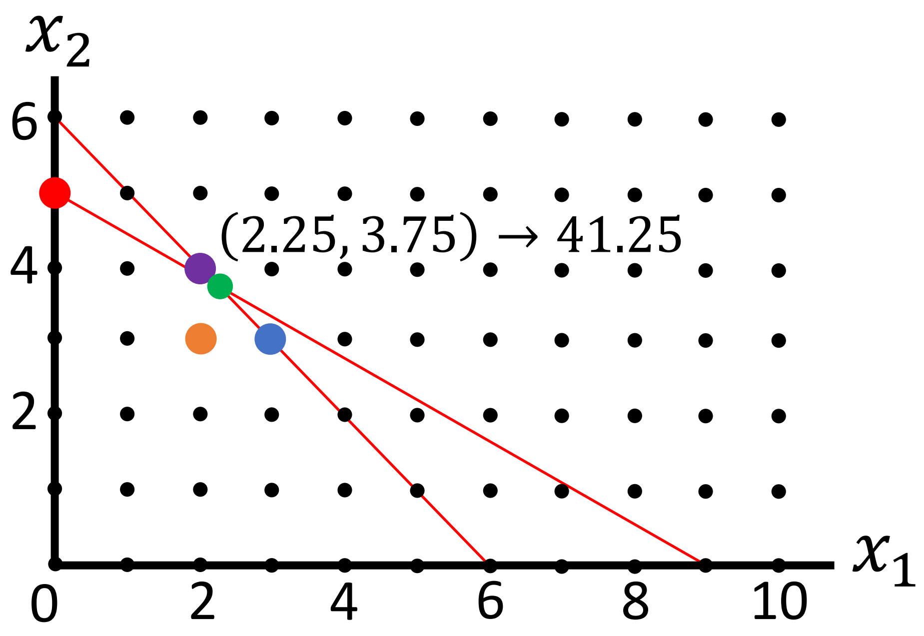
- Closest feasible integer solution? – Obj = 34
- Closest feasible integer solution on feasible region boundary? – Obj = 39

Why are ILPs hard to solve?



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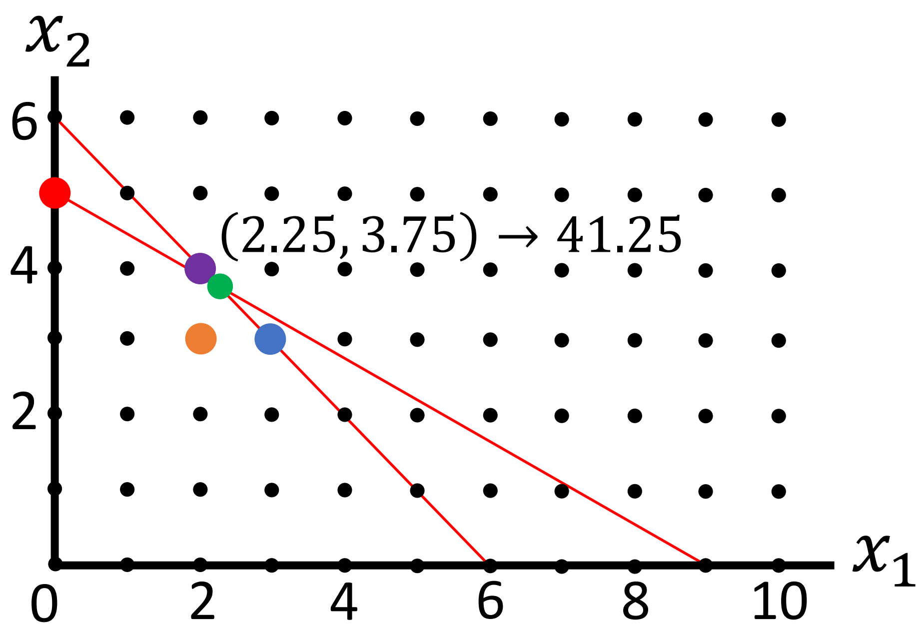
Optimal continuous solution → optimal integer solution?

- Closest integer solution? – Not feasible
- Closest feasible integer solution? – Obj = 34
- Closest feasible integer solution on feasible region boundary? – Obj = 39
- **Actual optimal – Obj = 40**

Why are ILPs hard to solve?



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Subject to:
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No guarantee that the optimal solution is on the feasible region boundary!

Optimal continuous solution → optimal integer solution?

- Closest integer solution? – Not feasible
- Closest feasible integer solution? – Obj = 34
- Closest feasible integer solution on feasible region boundary? – Obj = 39
- **Actual optimal – Obj = 40**

Why are ILPs hard to solve?

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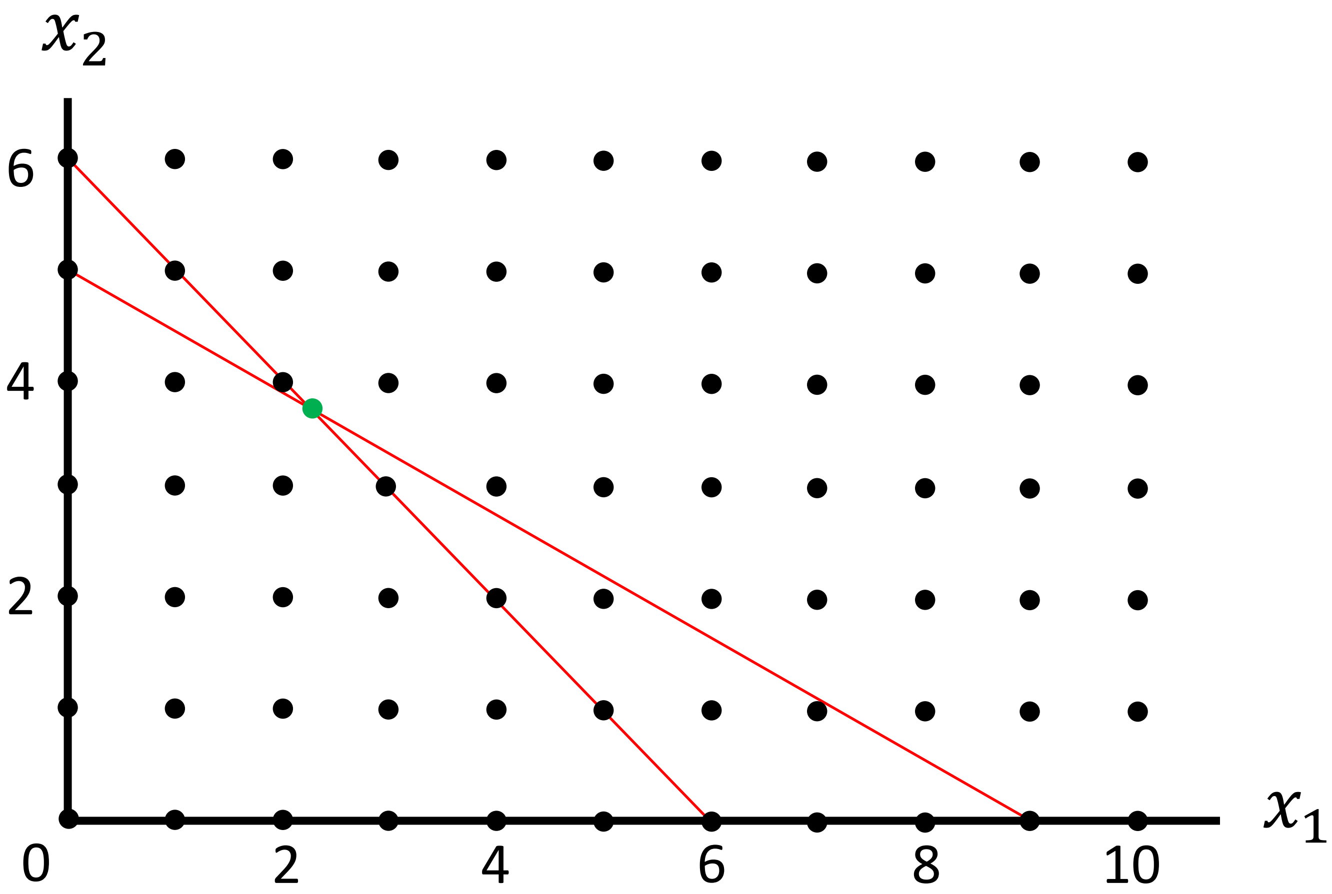

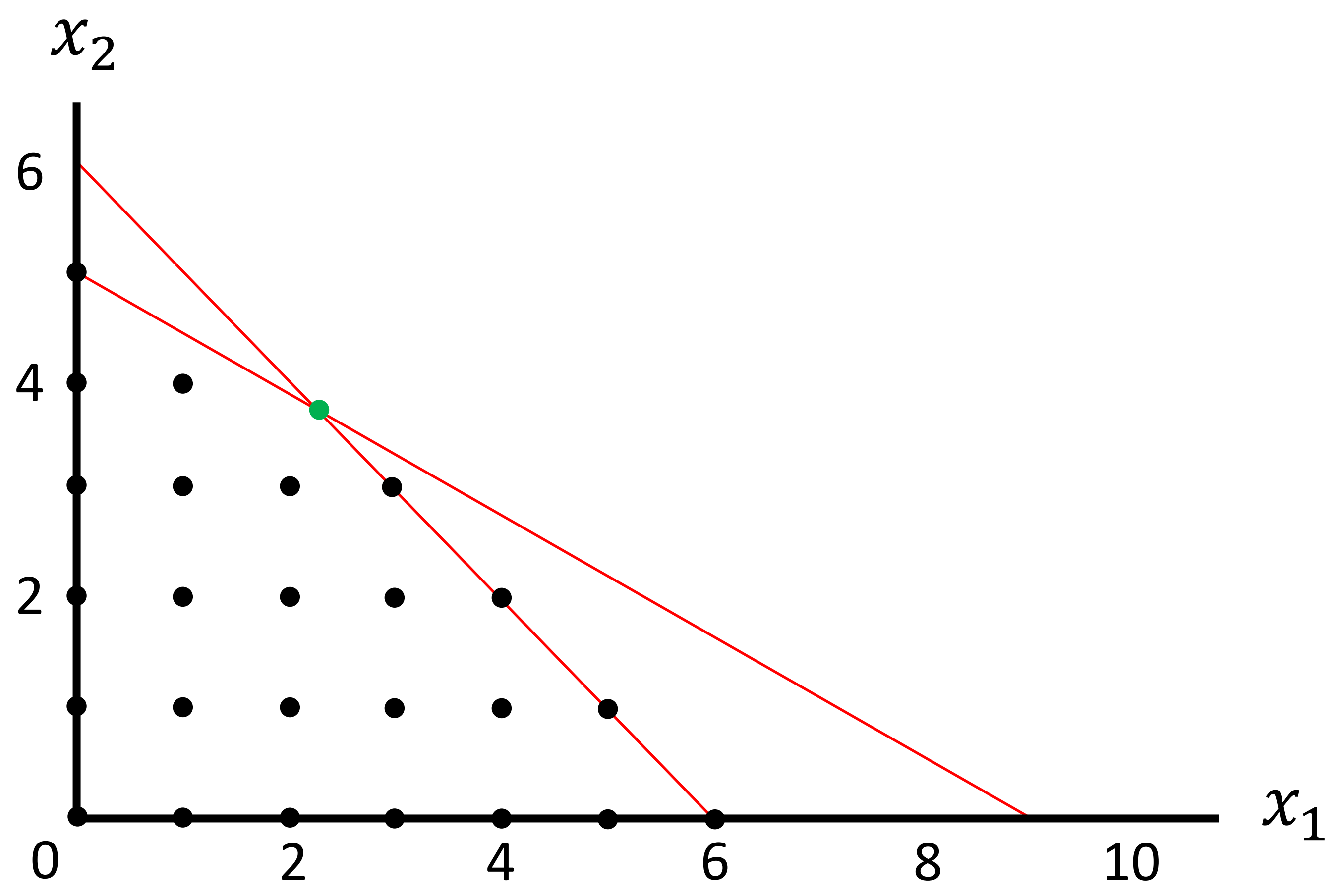

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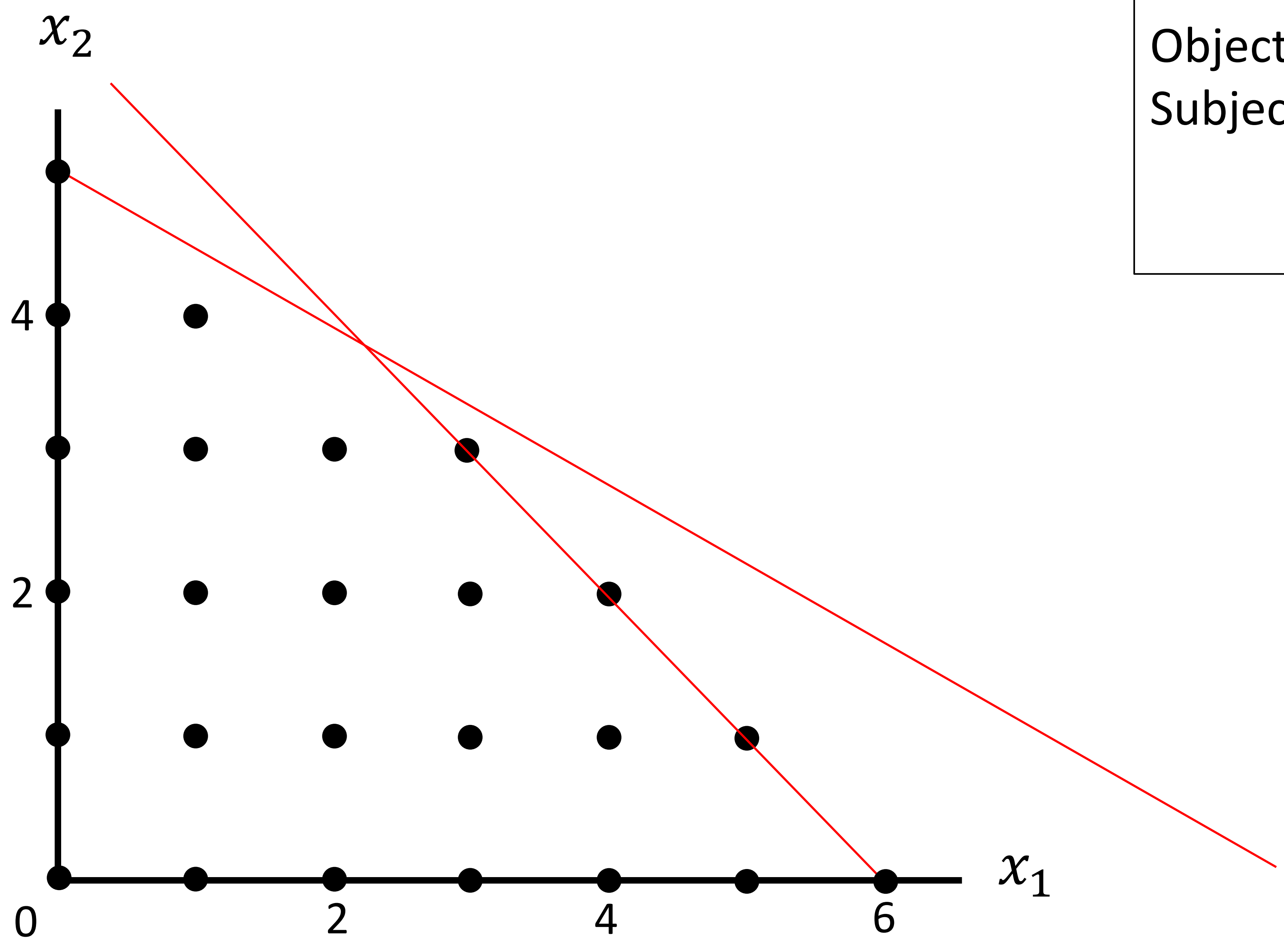

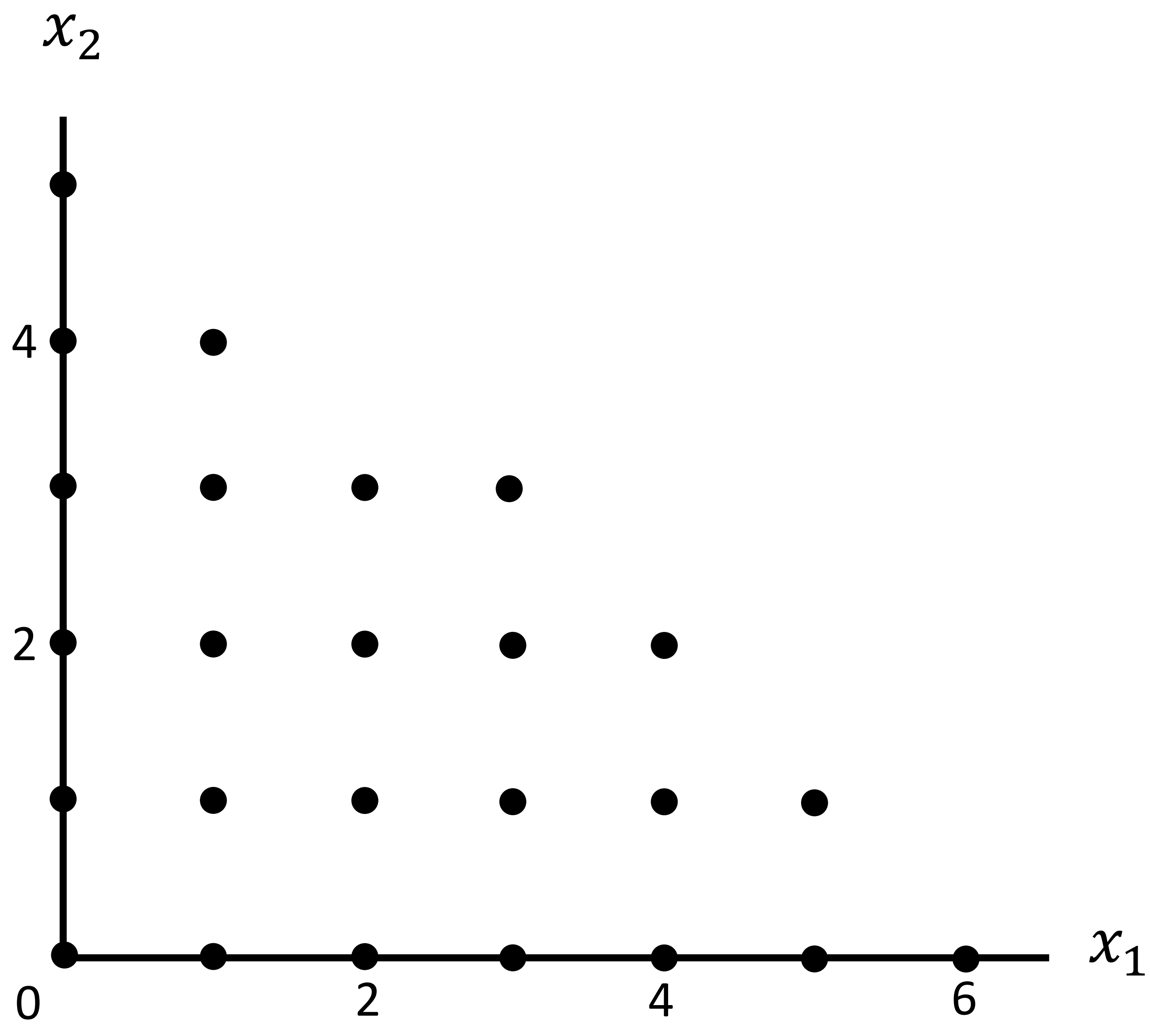

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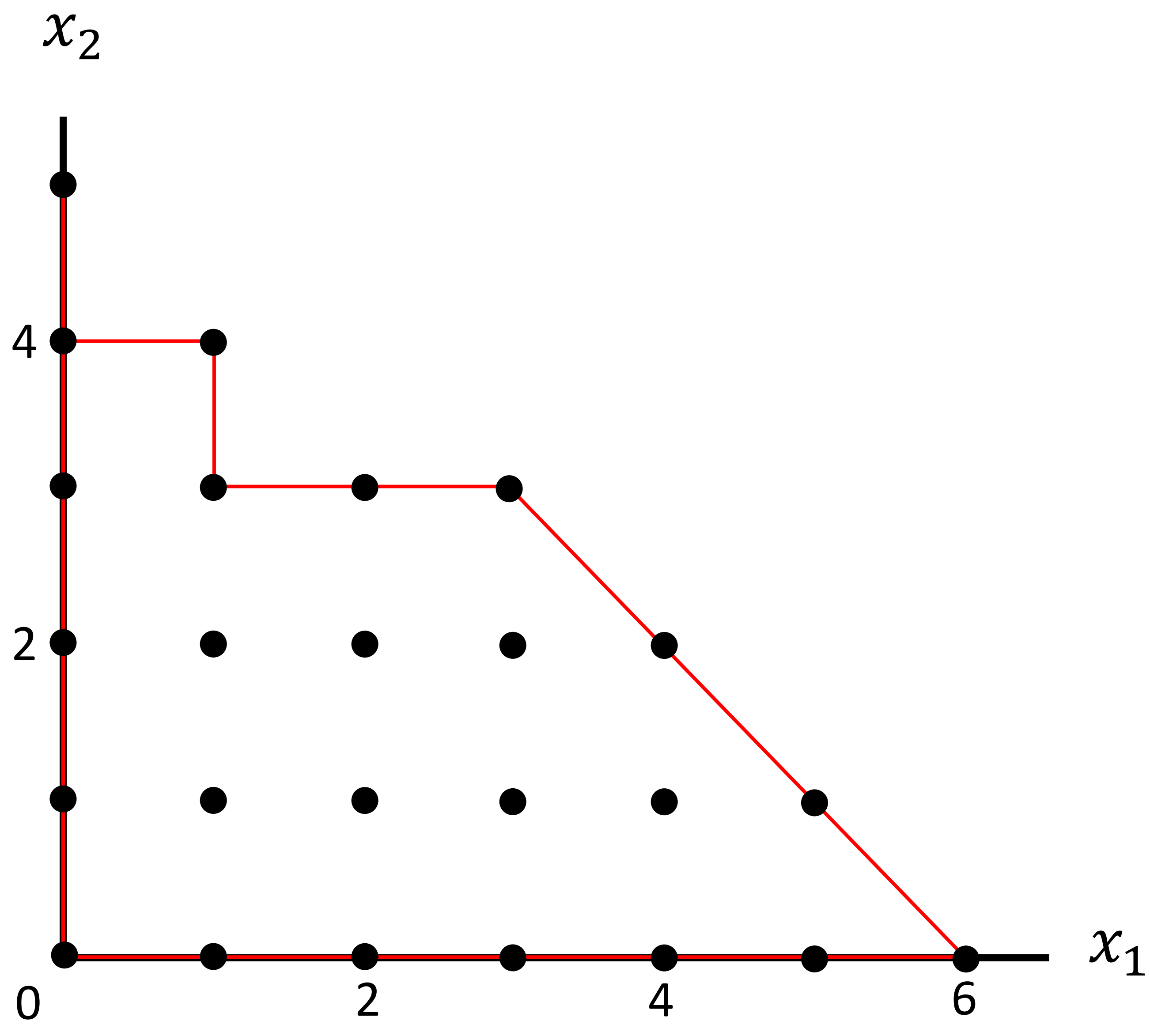
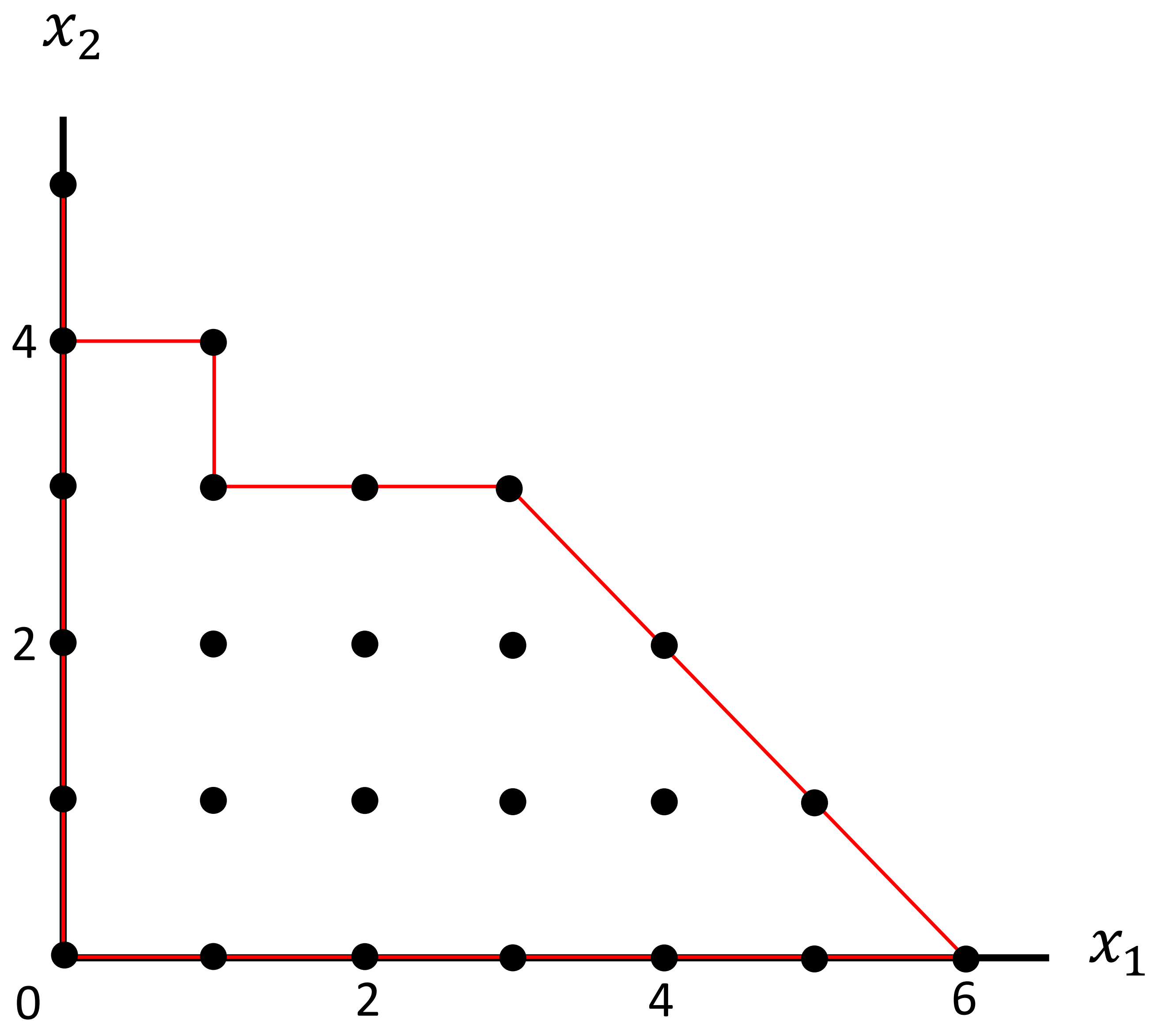


Integer feasible region:



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Integer feasible region:

- Not convex.

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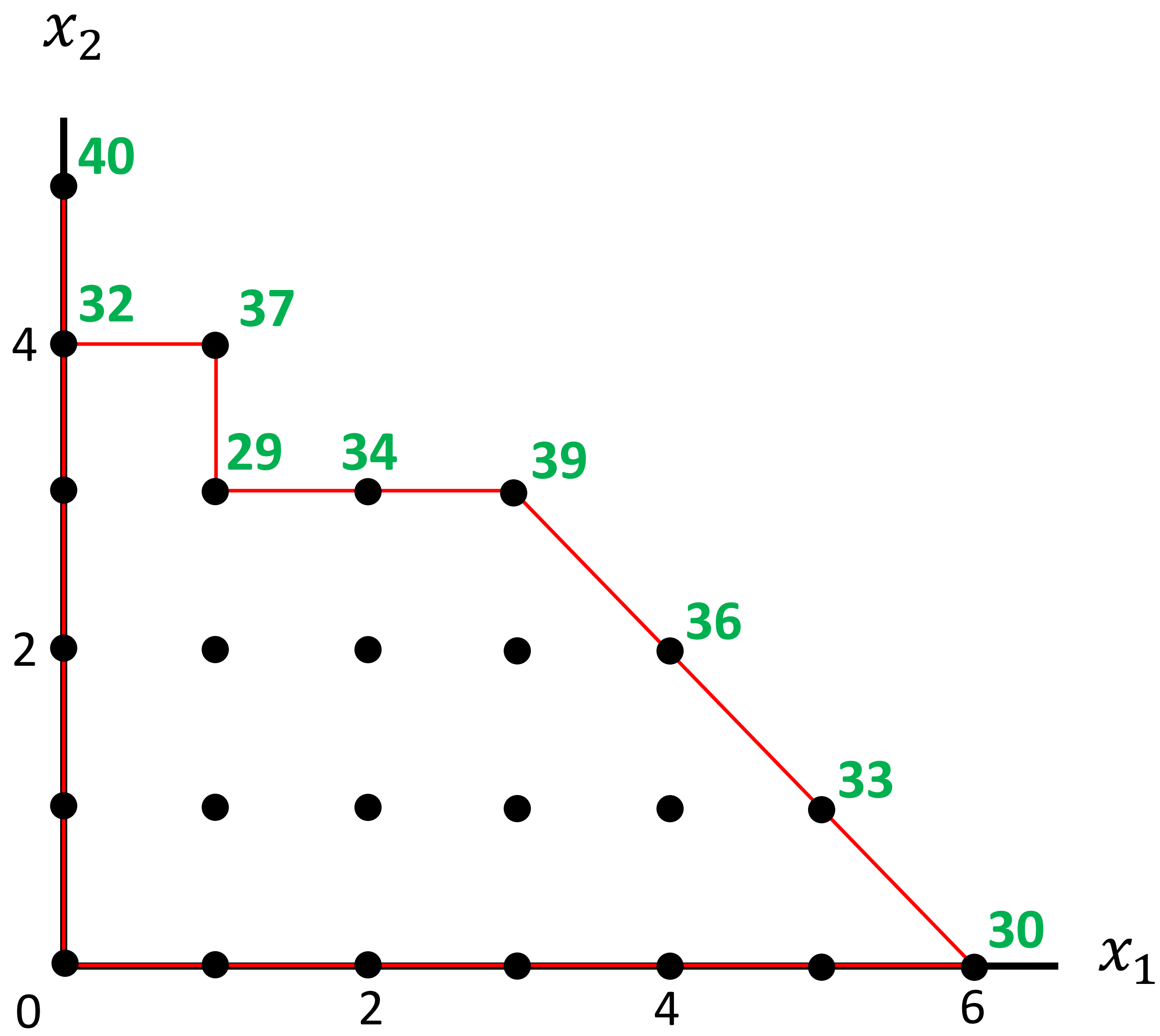

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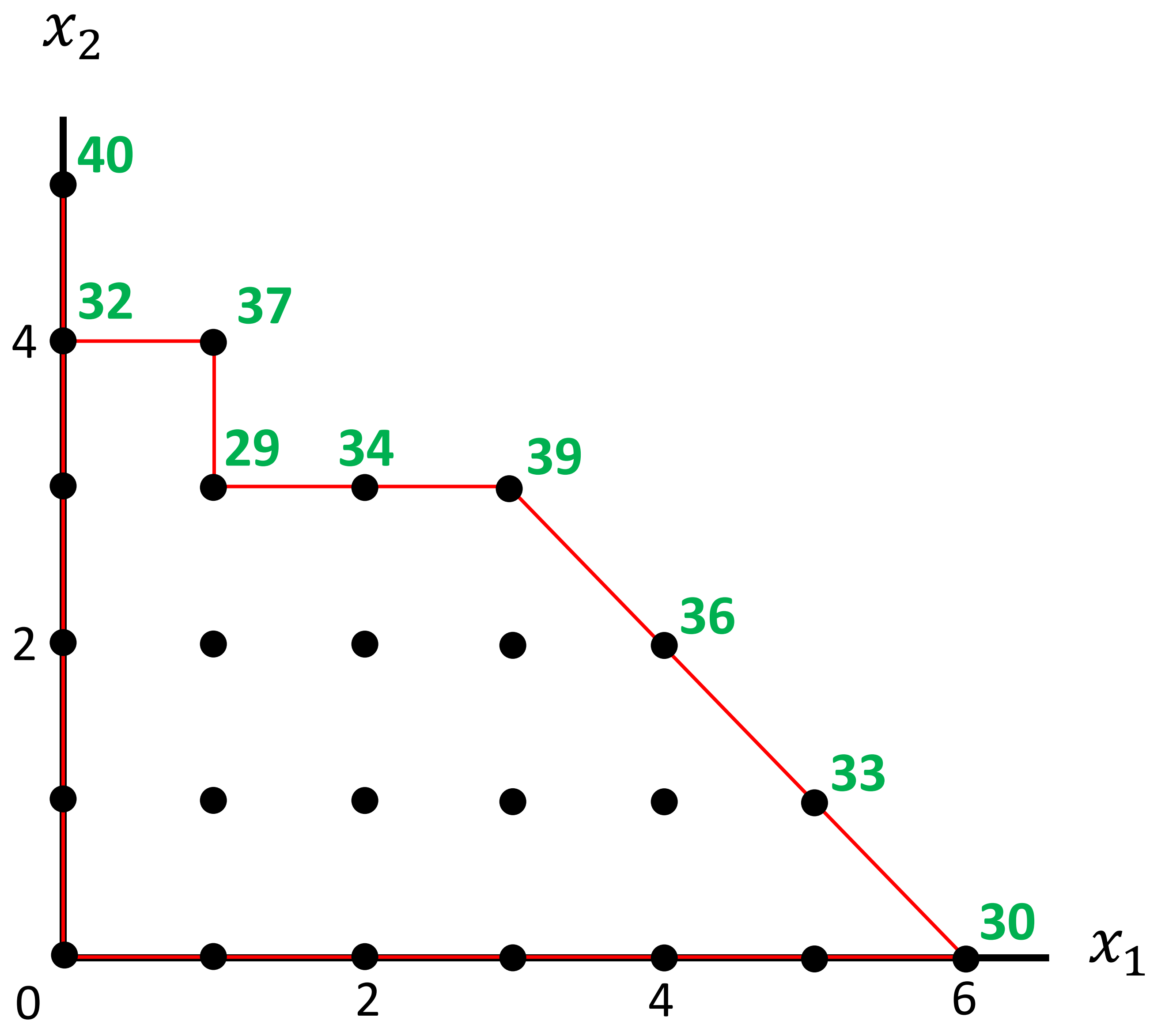

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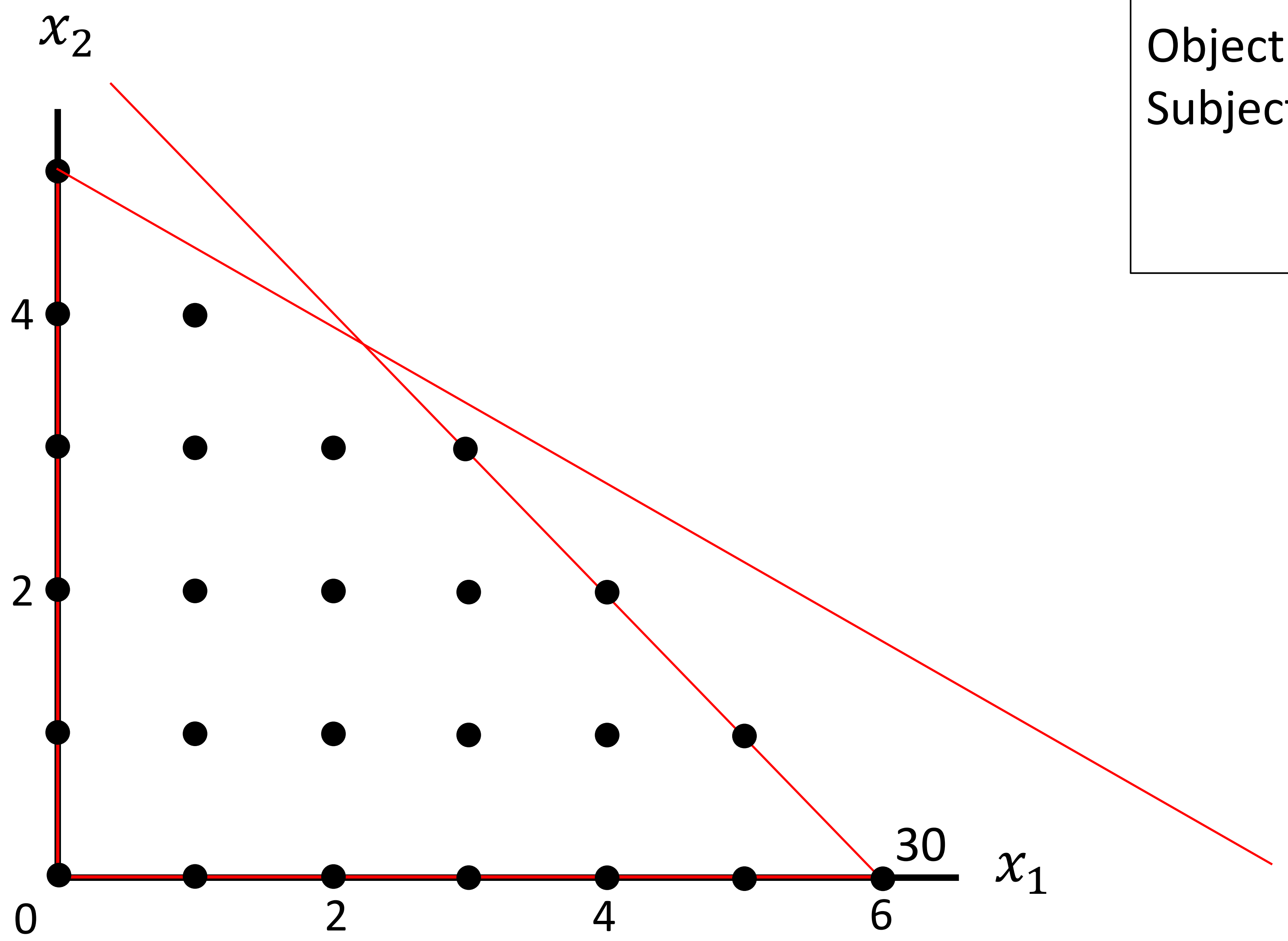
Integer feasible region:

- Not convex.
- local optimum $\neq$ global optimum.



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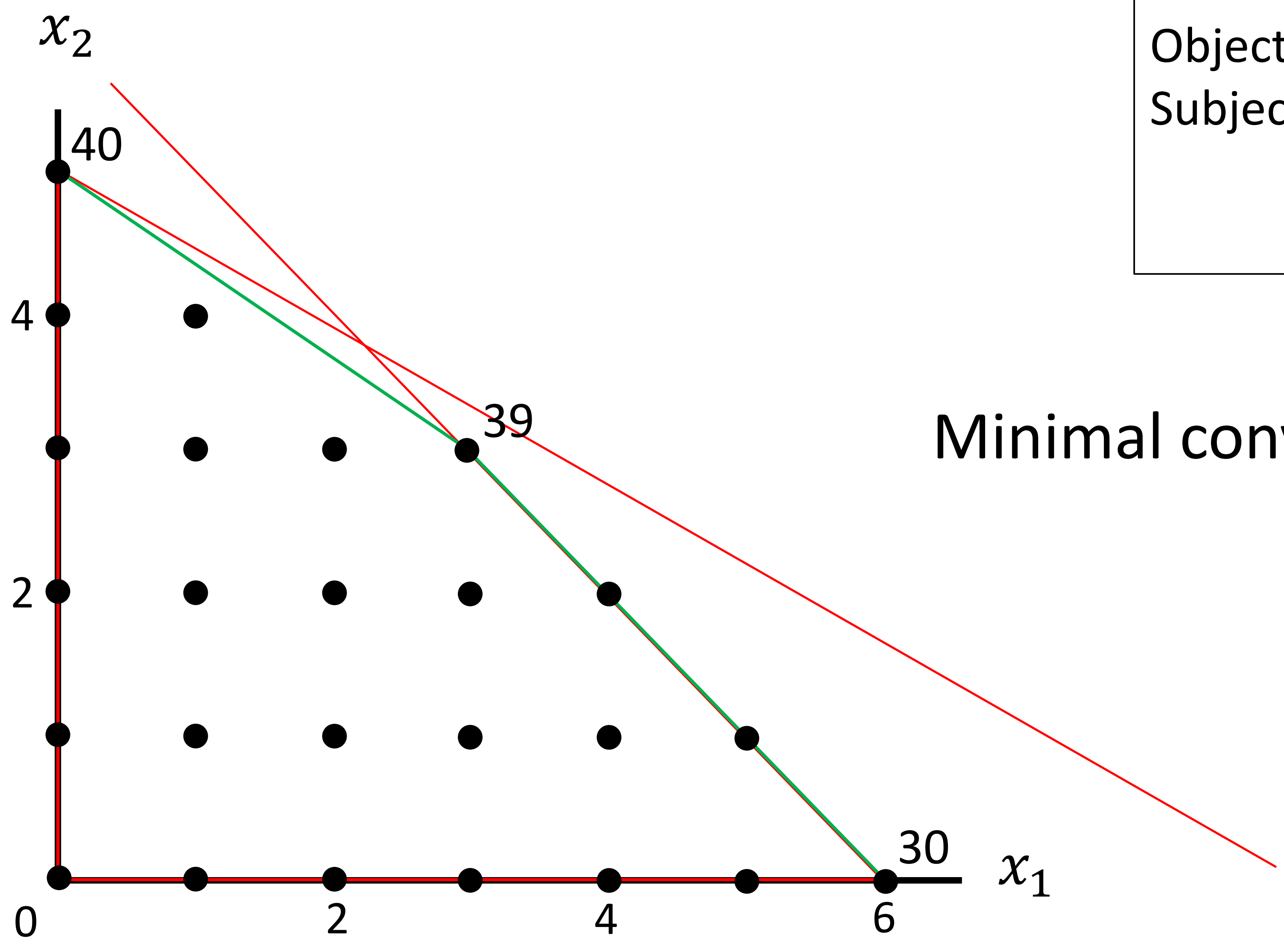

Minimal convex hull (integer hull):

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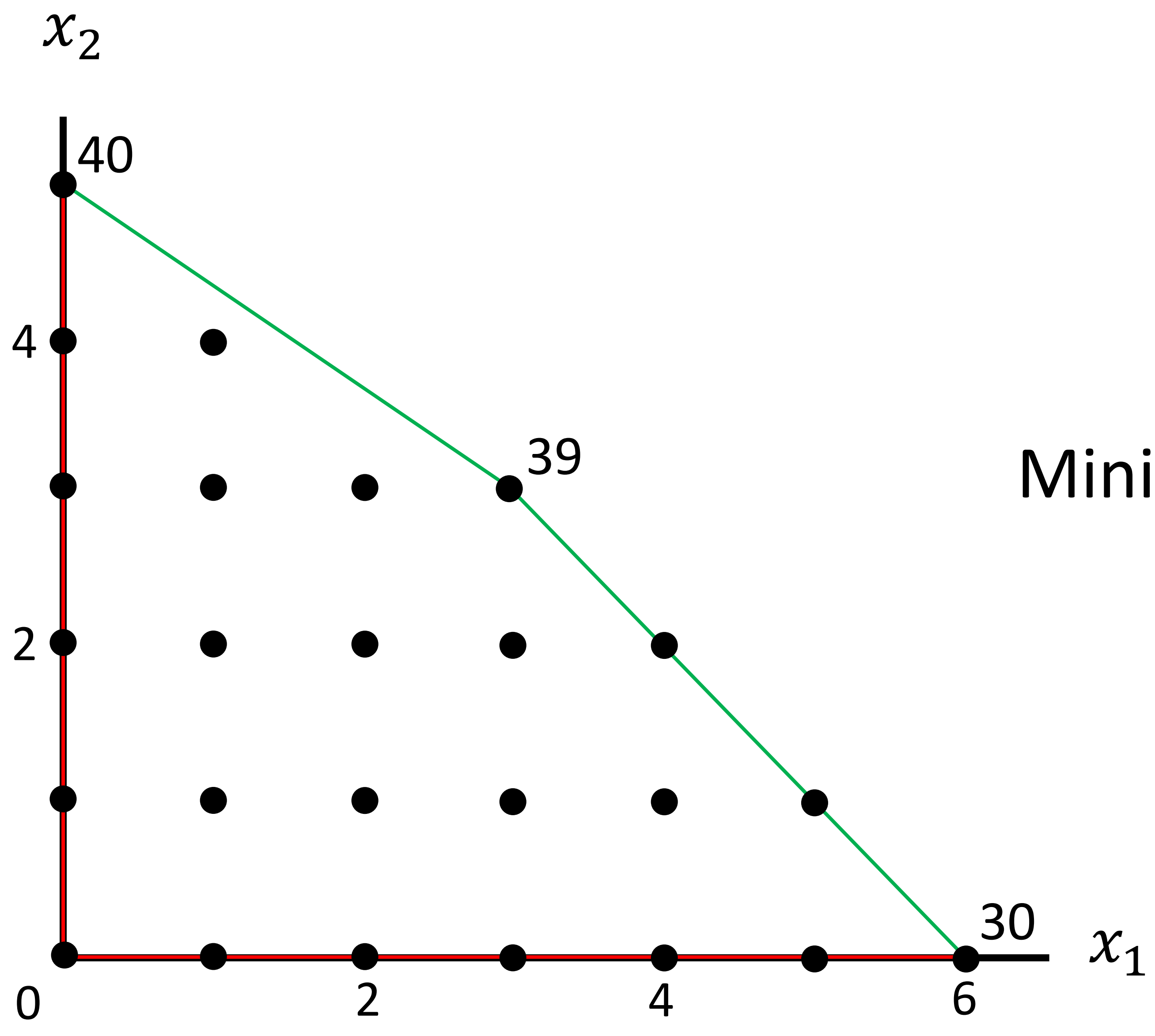

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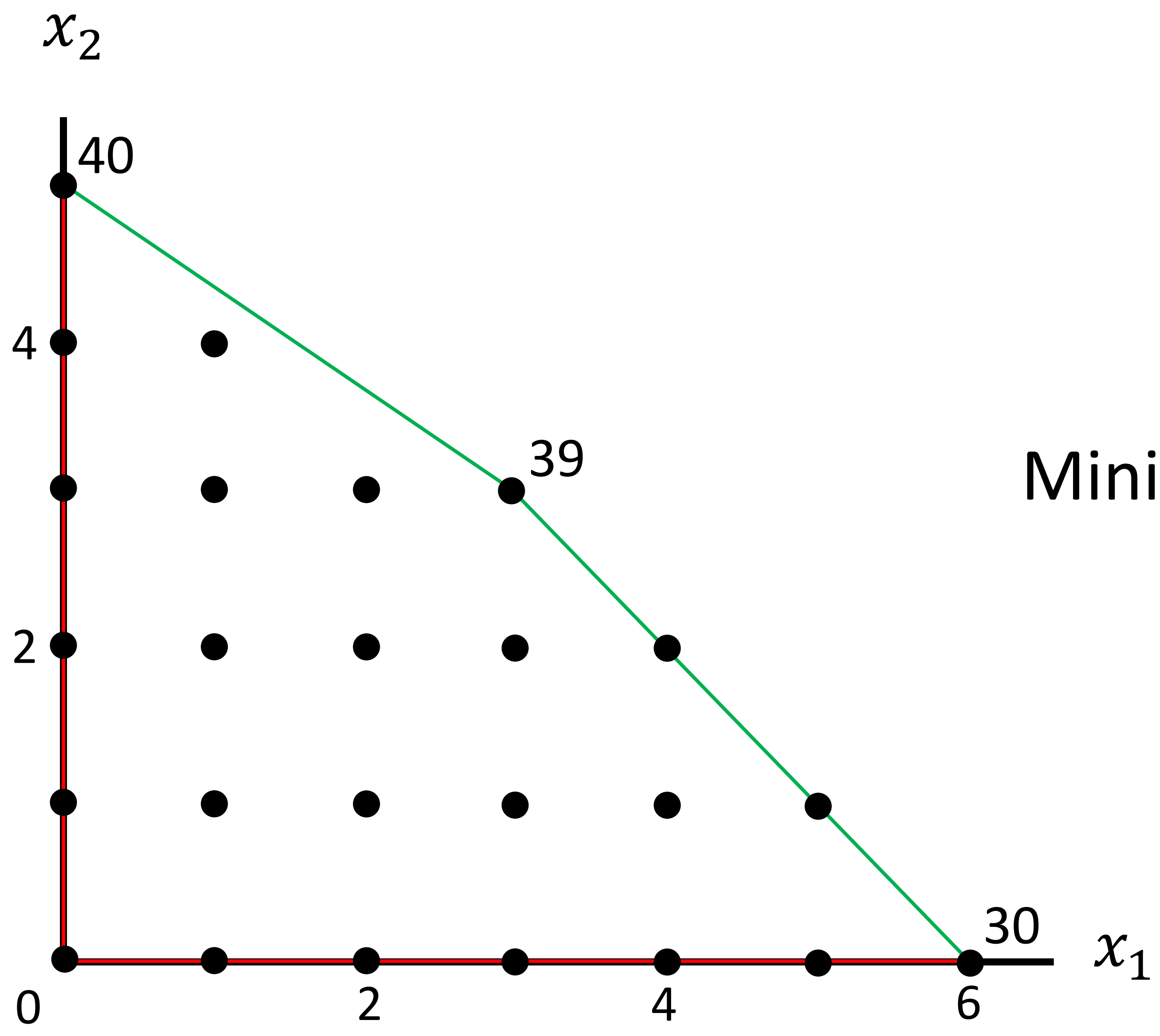

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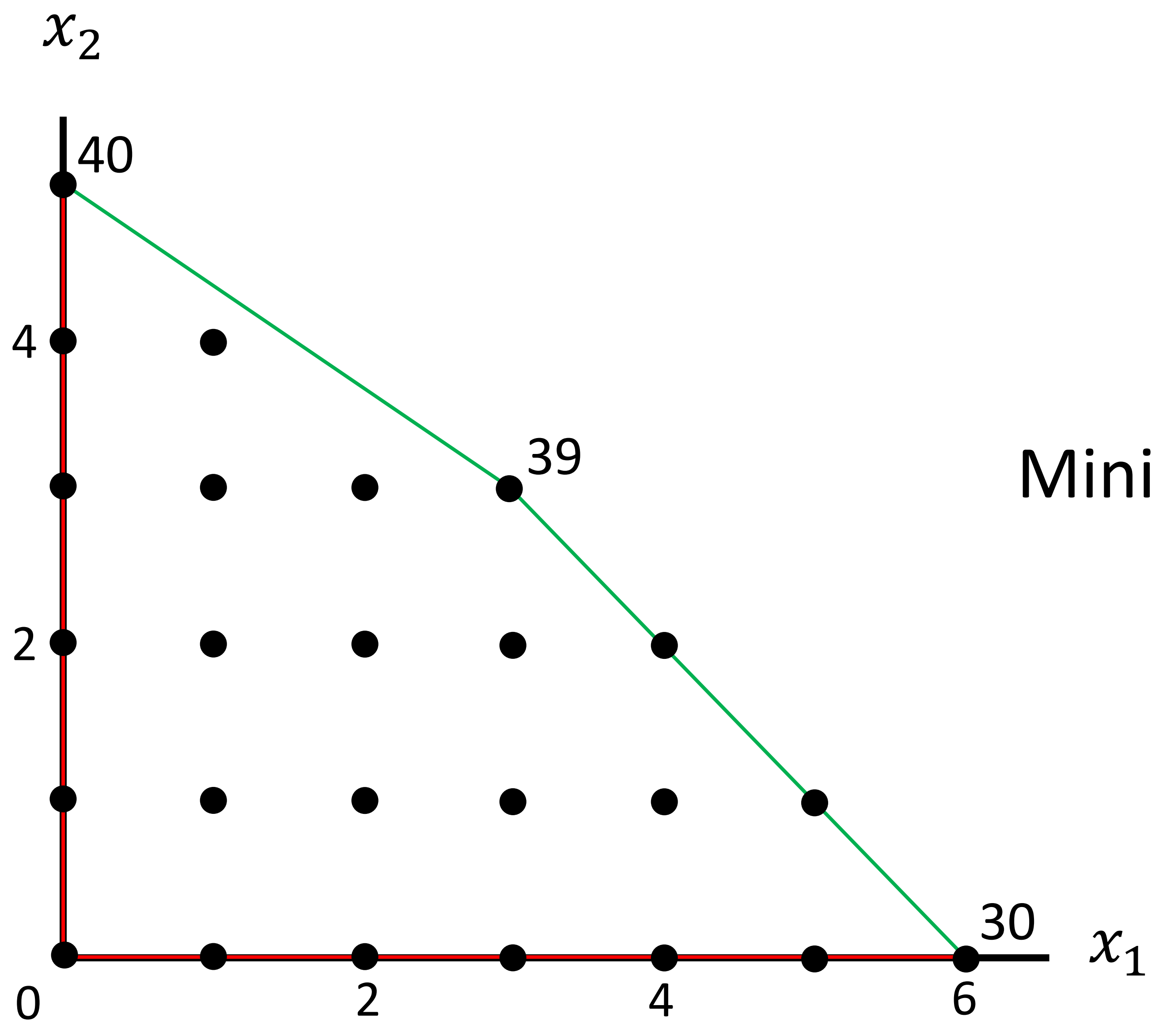

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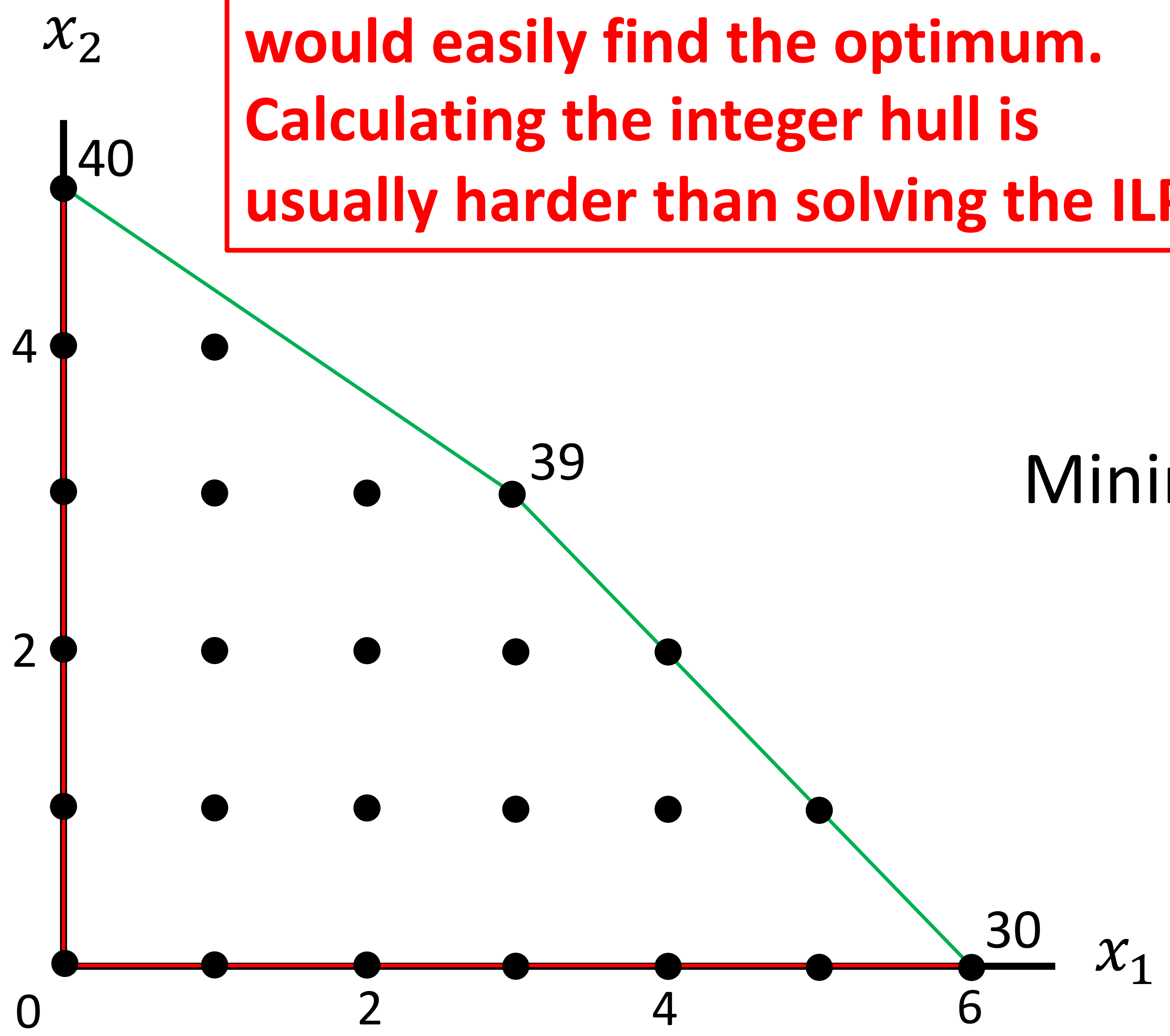
Minimal convex hull (integer hull):

- Convex.
- local optimum = global optimum.
- $O\left(n^{\lfloor d/2 \rfloor}\right)$ faces,

  $n = \#$ points

  $d = \#$ dimensions



If you had the integer hull, Simplex would easily find the optimum. Calculating the integer hull is usually harder than solving the ILP.

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$5x_1 + 9x_2 \leq 45$

$x_1, x_2 \geq 0$

Minimal convex hull (integer hull):

- Convex.
- local optimum $=$ global optimum.
- $O\left(n^{\lfloor d/2 \rfloor}\right)$ faces,

  $n = \#$ points

  $d = \#$ dimensions

Solving ILPs

How can we solve ILPs?

Solving ILPs

ILP

Maximization

Solving ILPs

ILP $\boldsymbol{x} \in \mathbb{Z} \to \boldsymbol{x} \in \mathbb{R}$ **LP Relaxation**

1. Relax ILP

Maximization

Solving ILPs

ILP $\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$ **LP Relaxation**

1. Relax ILP

Maximization

$x_i \in \{0,1\}$ = Indicates if vertex i is selected.

Objective: $\min \sum_i x_i$

Subject to: $x_i + x_j \geq 1$, for each edge $e = (i, j)$

$x_i \in [0,1]$ = Indicates if vertex i is selected.

Objective: $\min \sum_i x_i$

Subject to: $x_i + x_j \geq 1$, for each edge $e = (i, j)$

Solving ILPs

2. Solve relaxed LP.

ILP $\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$ **LP Relaxation**

1. Relax ILP

Maximization

Solving ILPs

2. Solve relaxed LP. Suppose the optimal objective is 7.4 with an integer solution (i.e., $x_i \in \mathbb{Z}, \forall i$).

ILP

$\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$

1. Relax ILP

LP Relaxation

$x \in \mathbb{Z}$, obj = 7.4

Maximization

What happens?

Solving ILPs

2. Solve relaxed LP. Suppose the optimal objective is 7.4 with an integer solution (i.e., $x_i \in \mathbb{Z}, \forall i$).

ILP $\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$ **LP Relaxation**

1. Relax ILP

Maximization

$\boldsymbol{x} \in \mathbb{Z}$**, obj = 7.4**

We've found the optimal!

Since the relaxed LP has **more options** to increase the objective value, $OPT_{ILP} \leq OPT_{LP}$ (for a maximization problem).

What happens?

Solving ILPs

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

ILP

$\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$

1. Relax ILP

LP Relaxation

$\boldsymbol{x} \in \mathbb{R}$, **obj = 100**

Maximization

UB = 100
LB = ?

Solving ILPs

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

ILP $\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$ **LP Relaxation**

1. Relax ILP

$\boldsymbol{x} \in \mathbb{R}$, **obj = 100**

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

Maximization

subproblem 1 **subproblem 2**

UB = 100
LB = ?

Solving ILPs

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

4. Solve subproblem LPs.

ILP $\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$ **LP Relaxation**

1. Relax ILP

$\boldsymbol{x} \in \mathbb{R}$, **obj = 100**

subproblem 1 **subproblem 2**

Maximization

UB = 100

LB = ?

Solving ILPs

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

4. Solve subproblem LPs.

ILP → $\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$ → **LP Relaxation**

1. Relax ILP

Maximization

LP Relaxation
$\boldsymbol{x} \in \mathbb{R}$, **obj = 100**

subproblem 1
$\boldsymbol{x} \in \mathbb{Z}$, **obj = 85**

subproblem 2
$\boldsymbol{x} \in \mathbb{Z}$, **obj = 90**

UB = 100
LB = ?

What happens?

Solving ILPs

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

ILP

$\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$

1. Relax ILP

LP Relaxation

$x \in \mathbb{R}$, **obj = 100**

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

Maximization

4. Solve subproblem LPs.

UB = 90
LB = 90

subproblem 1
$x \in \mathbb{Z}$, **obj = 85**

subproblem 2
$x \in \mathbb{Z}$, **obj = 90**

Optimal

What happens?

Solving ILPs

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

4. Solve subproblem LPs.

ILP

$\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$

1. Relax ILP

Maximization

LP Relaxation

$x \in \mathbb{R}$, **obj = 100**

subproblem 1

$x \in \mathbb{R}$, **obj = 98**

subproblem 2

$x \in \mathbb{Z}$, **obj = 90**

UB = 98
LB = 90

Solving ILPs

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

ILP $\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$ **LP Relaxation**

1. Relax ILP

$\boldsymbol{x} \in \mathbb{R}$, **obj = 100**

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

Maximization

4. Solve subproblem LPs.

UB = 98
LB = 90

subproblem 1
$\boldsymbol{x} \in \mathbb{R}$, **obj = 98**

subproblem 2
$\boldsymbol{x} \in \mathbb{Z}$, **obj = 90**

So far, we have:

1. Feasible (integer) solution.

Solving ILPs

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

4. Solve subproblem LPs.

ILP $\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$ **LP Relaxation**

1. Relax ILP

Maximization

LP Relaxation: $\boldsymbol{x} \in \mathbb{R}$, **obj = 100**

subproblem 1: $\boldsymbol{x} \in \mathbb{R}$, **obj = 98**

subproblem 2: $\boldsymbol{x} \in \mathbb{Z}$, **obj = 90**

UB = 98
LB = 90

So far, we have:

1. Feasible (integer) solution.
2. Upper and lower bounds on optimal.

Solving ILPs

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

ILP $\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$ **LP Relaxation**

1. Relax ILP

$\boldsymbol{x} \in \mathbb{R}$, **obj = 100**

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

Maximization

4. Solve subproblem LPs.

UB = 98
LB = 90

subproblem 1
$\boldsymbol{x} \in \mathbb{R}$, **obj = 98**

subproblem 2
$\boldsymbol{x} \in \mathbb{Z}$, **obj = 90**

So far, we have:

1. Feasible (integer) solution.
2. Upper and lower bounds on optimal.

Branch and Bound Plan: Use these to restrict the search space and identify optimality.

Solving ILPs

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

4. Solve subproblem LPs.

5. Formulate two new LPs that split the subproblem on some non-integer variable.

ILP

$\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$

1. Relax ILP

Maximization

UB = 98

LB = 90

LP Relaxation

$\boldsymbol{x} \in \mathbb{R}$, **obj = 100**

subproblem 1

$\boldsymbol{x} \in \mathbb{R}$, **obj = 98**

subproblem 2

$\boldsymbol{x} \in \mathbb{Z}$, **obj = 90**

subproblem 1.1

subproblem 1.2

Solving ILPs

ILP

$\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$

1. Relax ILP

Maximization

UB = 98
LB = 90

LP Relaxation
$\boldsymbol{x} \in \mathbb{R}$, **obj = 100**

subproblem 1
$\boldsymbol{x} \in \mathbb{R}$, **obj = 98**

subproblem 2
$\boldsymbol{x} \in \mathbb{Z}$, **obj = 90**

subproblem 1.1

subproblem 1.2

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

4. Solve subproblem LPs.

5. Formulate two new LPs that split the subproblem on some non-integer variable.

6. Solve the subproblem LPs.

Solving ILPs

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

4. Solve subproblem LPs.

5. Formulate two new LPs that split the subproblem on some non-integer variable.

6. Solve the subproblem LPs.

ILP $\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$ **LP Relaxation**

1. Relax ILP

Maximization

LP Relaxation
$\boldsymbol{x} \in \mathbb{R}$, **obj = 100**

subproblem 1
$\boldsymbol{x} \in \mathbb{R}$, **obj = 98**

subproblem 2
$\boldsymbol{x} \in \mathbb{Z}$, **obj = 90**

subproblem 1.1
$\boldsymbol{x} \in \mathbb{R}$, **obj = 88**

subproblem 1.2
$\boldsymbol{x} \in \mathbb{R}$, **obj = 97**

UB = ?
LB = ?

What happens?

Solving ILPs

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

ILP $\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$ **LP Relaxation**

1. Relax ILP

Maximization

$x \in \mathbb{R}$, **obj = 100**

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

4. Solve subproblem LPs.

UB = 97
LB = 90

subproblem 1
$x \in \mathbb{R}$, **obj = 98**

subproblem 2
$x \in \mathbb{Z}$, **obj = 90**

5. Formulate two new LPs that split the subproblem on some non-integer variable.

6. Solve the subproblem LPs.

~~subproblem 1.1~~
$x \in \mathbb{R}$, **obj = 88**

subproblem 1.2
$x \in \mathbb{R}$, **obj = 97**

What happens?

subproblem 1.1 is a dead end.
Prune it and keep going.

Solving ILPs

ILP

$\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$

1. Relax ILP

Maximization

LP Relaxation

$\boldsymbol{x} \in \mathbb{R}$, obj = 100

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

4. Solve subproblem LPs.

UB = ?

LB = ?

subproblem 1

$\boldsymbol{x} \in \mathbb{R}$, obj = 98

subproblem 2

$\boldsymbol{x} \in \mathbb{Z}$, obj = 90

5. Formulate two new LPs that split the subproblem on some non-integer variable.

subproblem 1.1

$\boldsymbol{x} \in \mathbb{R}$, obj = 88

subproblem 1.2

$\boldsymbol{x} \in \mathbb{R}$, obj = 85

6. Solve the subproblem LPs.

What happens?

Solving ILPs

ILP

$\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$

1. Relax ILP

LP Relaxation

$x \in \mathbb{R}$, obj = 100

Maximization

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

4. Solve subproblem LPs.

5. Formulate two new LPs that split the subproblem on some non-integer variable.

6. Solve the subproblem LPs.

UB = 90

LB = 90

subproblem 1

$x \in \mathbb{R}$, obj = 98

subproblem 2

$x \in \mathbb{Z}$, obj = 90

~~subproblem 1.1~~

$x \in \mathbb{R}$, obj = 88

~~subproblem 1.2~~

$x \in \mathbb{R}$, obj = 85

Optimal

What happens?

Solving ILPs

ILP

$\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$

1. Relax ILP

Maximization

LP Relaxation

$x \in \mathbb{R}$, obj = 100

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

UB = 97
LB = 90

subproblem 1

$x \in \mathbb{R}$, obj = 98

subproblem 2

$x \in \mathbb{Z}$, obj = 90

~~subproblem 1.1~~

$x \in \mathbb{R}$, obj = 88

subproblem 1.2

$x \in \mathbb{R}$, obj = 97

4. Solve subproblem LPs.

5. Formulate two new LPs that split the subproblem on some non-integer variable.

6. Solve the subproblem LPs.

7. Repeat.

Solving ILPs

ILP — $\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$ — **LP Relaxation**

1. Relax ILP

Maximization

UB = 97
LB = 90

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

4. Solve subproblem LPs.

5. Formulate two new LPs that split the subproblem on some non-integer variable.

6. Solve the subproblem LPs.

7. Repeat.

LP Relaxation
$\boldsymbol{x} \in \mathbb{R}$, **obj = 100**

- **subproblem 1**
  $\boldsymbol{x} \in \mathbb{R}$, **obj = 98**
  - ~~**subproblem 1.1**~~
    $\boldsymbol{x} \in \mathbb{R}$, **obj = 88**
  - **subproblem 1.2**
    $\boldsymbol{x} \in \mathbb{R}$, **obj = 97**
    - **subproblem 1.2.1**
    - **subproblem 1.2.2**
- **subproblem 2**
  $\boldsymbol{x} \in \mathbb{Z}$, **obj = 90**

Solving ILPs

ILP

Maximization

$\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$

1. Relax ILP

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

4. Solve subproblem LPs.

5. Formulate two new LPs that split the subproblem on some non-integer variable.

6. Solve the subproblem LPs.

7. Repeat.

UB = ?

LB = ?

LP Relaxation
$x \in \mathbb{R}$, obj = 100

subproblem 1
$x \in \mathbb{R}$, obj = 98

subproblem 2
$x \in \mathbb{Z}$, obj = 90

~~subproblem 1.1~~
$x \in \mathbb{R}$, obj = 88

subproblem 1.2
$x \in \mathbb{R}$, obj = 97

subproblem 1.2.1
$x \in \mathbb{R}$, obj = 94

subproblem 1.2.2
$x \in \mathbb{Z}$, obj = 95

What happens?

Solving ILPs

ILP

$\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$

1. Relax ILP

Maximization

LP Relaxation

$x \in \mathbb{R}$, obj = 100

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

4. Solve subproblem LPs.

5. Formulate two new LPs that split the subproblem on some non-integer variable.

6. Solve the subproblem LPs.

7. Repeat.

UB = 95
LB = 95

subproblem 1
$x \in \mathbb{R}$, obj = 98

subproblem 2
$x \in \mathbb{Z}$, obj = 90

~~subproblem 1.1~~
$x \in \mathbb{R}$, obj = 88

subproblem 1.2
$x \in \mathbb{R}$, obj = 97

~~subproblem 1.2.1~~
$x \in \mathbb{R}$, obj = 94

subproblem 1.2.2
$x \in \mathbb{Z}$, obj = 95

Optimal

What happens?

Solving ILPs

ILP

$\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$

1. Relax ILP

Maximization

UB = ?
LB = ?

LP Relaxation

$\boldsymbol{x} \in \mathbb{R}$, **obj = 100**

subproblem 1

$\boldsymbol{x} \in \mathbb{R}$, **obj = 98**

subproblem 2

$\boldsymbol{x} \in \mathbb{Z}$, **obj = 90**

~~**subproblem 1.1**~~

$\boldsymbol{x} \in \mathbb{R}$, **obj = 88**

subproblem 1.2

$\boldsymbol{x} \in \mathbb{R}$, **obj = 97**

subproblem 1.2.1

$\boldsymbol{x} \in \mathbb{R}$, **obj = 85**

subproblem 1.2.2

$\boldsymbol{x} \in \mathbb{Z}$, **obj = 89**

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

4. Solve subproblem LPs.

5. Formulate two new LPs that split the subproblem on some non-integer variable.

6. Solve the subproblem LPs.

7. Repeat.

What happens?

Solving ILPs

ILP $\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$ **LP Relaxation**

1. Relax ILP

Maximization

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

4. Solve subproblem LPs.

5. Formulate two new LPs that split the subproblem on some non-integer variable.

6. Solve the subproblem LPs.

7. Repeat.

LP Relaxation
$x \in \mathbb{R}$, obj = 100

subproblem 1
$x \in \mathbb{R}$, obj = 98

subproblem 2
$x \in \mathbb{Z}$, obj = 90

Optimal

~~subproblem 1.1~~
$x \in \mathbb{R}$, obj = 88

subproblem 1.2
$x \in \mathbb{R}$, obj = 97

~~subproblem 1.2.1~~
$x \in \mathbb{R}$, obj = 85

subproblem 1.2.2
$x \in \mathbb{Z}$, obj = 89

UB = 90
LB = 90

What happens?

Solving ILPs

ILP

$\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$

1. Relax ILP

Maximization

UB = ?

LB = ?

LP Relaxation

$x \in \mathbb{R}$, obj = 100

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

4. Solve subproblem LPs.

5. Formulate two new LPs that split the subproblem on some non-integer variable.

6. Solve the subproblem LPs.

7. Repeat.

subproblem 1

$x \in \mathbb{R}$, obj = 98

subproblem 2

$x \in \mathbb{Z}$, obj = 90

~~subproblem 1.1~~

$x \in \mathbb{R}$, obj = 88

subproblem 1.2

$x \in \mathbb{R}$, obj = 97

subproblem 1.2.1

$x \in \mathbb{R}$, obj = 89

subproblem 1.2.2

$x \in \mathbb{R}$, obj = 88

What happens?

Solving ILPs

ILP

Maximization

$\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$

1. Relax ILP

LP Relaxation

$x \in \mathbb{R}$, obj = 100

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

4. Solve subproblem LPs.

5. Formulate two new LPs that split the subproblem on some non-integer variable.

6. Solve the subproblem LPs.

7. Repeat.

UB = 90
LB = 90

subproblem 1
$x \in \mathbb{R}$, obj = 98

subproblem 2
$x \in \mathbb{Z}$, obj = 90

Optimal

~~subproblem 1.1~~
$x \in \mathbb{R}$, obj = 88

subproblem 1.2
$x \in \mathbb{R}$, obj = 97

~~subproblem 1.2.1~~
$x \in \mathbb{R}$, obj = 89

~~subproblem 1.2.2~~
$x \in \mathbb{R}$, obj = 88

What happens?

Solving ILPs

ILP

$\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$

1. Relax ILP

Maximization

LP Relaxation

$x \in \mathbb{R}$, obj = 100

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

UB = ?

LB = ?

subproblem 1

$x \in \mathbb{R}$, obj = 98

subproblem 2

$x \in \mathbb{Z}$, obj = 90

4. Solve subproblem LPs.

5. Formulate two new LPs that split the subproblem on some non-integer variable.

~~subproblem 1.1~~

$x \in \mathbb{R}$, obj = 88

subproblem 1.2

$x \in \mathbb{R}$, obj = 97

6. Solve the subproblem LPs.

7. Repeat.

What happens?

subproblem 1.2.1

$x \in \mathbb{R}$, obj = 94

subproblem 1.2.2

$x \in \mathbb{Z}$, obj = 88

Solving ILPs

ILP $x \in \mathbb{Z} \rightarrow x \in \mathbb{R}$ **LP Relaxation**

1. Relax ILP

Maximization

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

4. Solve subproblem LPs.

5. Formulate two new LPs that split the subproblem on some non-integer variable.

6. Solve the subproblem LPs.

7. Repeat.

UB = 94
LB = 90

LP Relaxation
$x \in \mathbb{R}$, **obj = 100**

subproblem 1
$x \in \mathbb{R}$, **obj = 98**

subproblem 2
$x \in \mathbb{Z}$, **obj = 90**

~~subproblem 1.1~~
$x \in \mathbb{R}$, **obj = 88**

subproblem 1.2
$x \in \mathbb{R}$, **obj = 97**

subproblem 1.2.1
$x \in \mathbb{R}$, **obj = 94**

subproblem 1.2.2
$x \in \mathbb{Z}$, **obj = 88**

What happens?

Need to continue. There could be an integer solution buried in subproblem 1.2.1 with a better objective value.

Solving ILPs

ILP

$x \in \mathbb{Z} \rightarrow x \in \mathbb{R}$

1. Relax ILP

Maximization

UB = ?

LB = ?

LP Relaxation

$x \in \mathbb{R}$, obj = 100

subproblem 1

$x \in \mathbb{R}$, obj = 98

subproblem 2

$x \in \mathbb{Z}$, obj = 90

~~**subproblem 1.1**~~

$x \in \mathbb{R}$, obj = 88

subproblem 1.2

$x \in \mathbb{R}$, obj = 97

subproblem 1.2.1

$x \in \mathbb{R}$, obj = 94

subproblem 1.2.2

$x \in \mathbb{Z}$, obj = 92

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

4. Solve subproblem LPs.

5. Formulate two new LPs that split the subproblem on some non-integer variable.

6. Solve the subproblem LPs.

7. Repeat.

What happens?

Solving ILPs

ILP → $x \in \mathbb{Z} \to x \in \mathbb{R}$ → **LP Relaxation**

1. Relax ILP

Maximization

UB = 94
LB = 92

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

4. Solve subproblem LPs.

5. Formulate two new LPs that split the subproblem on some non-integer variable.

6. Solve the subproblem LPs.

7. Repeat.

LP Relaxation
$x \in \mathbb{R}$, **obj = 100**

subproblem 1
$x \in \mathbb{R}$, **obj = 98**

subproblem 2
$x \in \mathbb{Z}$, **obj = 90**

~~**subproblem 1.1**~~
$x \in \mathbb{R}$, **obj = 88**

subproblem 1.2
$x \in \mathbb{R}$, **obj = 97**

subproblem 1.2.1
$x \in \mathbb{R}$, **obj = 94**

subproblem 1.2.2
$x \in \mathbb{Z}$, **obj = 92**

What happens?

Need to continue. There could be an integer solution buried in subproblem 1.2.1 with a better objective value.

Solving ILPs

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

4. Solve subproblem LPs.

5. Formulate two new LPs that split the subproblem on some non-integer variable.

6. Solve the subproblem LPs.

7. Repeat.

ILP $\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$ **LP Relaxation**

1. Relax ILP

Maximization

UB = 90

LB = 90

LP Relaxation
$x \in \mathbb{R}$, **obj = 100**

subproblem 1
$x \in \mathbb{R}$, **obj = 98**

subproblem 2
$x \in \mathbb{Z}$, **obj = 90**

Optimal

~~**subproblem 1.1**~~
$x \in \mathbb{R}$, **obj = 88**

subproblem 1.2
$x \in \mathbb{R}$, **obj = 97**

~~**subproblem 1.2.1**~~
$x \in \mathbb{R}$, **obj = 85**

~~**subproblem 1.2.2**~~
$x \in \mathbb{Z}$, **obj = 89**

Solving ILPs

ILP

Maximization

$\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$

1. Relax ILP

LP Relaxation

$x \in \mathbb{R}$, obj = 100

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

4. Solve subproblem LPs.

5. Formulate two new LPs that split the subproblem on some non-integer variable.

6. Solve the subproblem LPs.

7. Repeat.

UB = 90

LB = 90

subproblem 1

$x \in \mathbb{R}$, obj = 98

subproblem 2

$x \in \mathbb{Z}$, obj = 90

Optimal

~~subproblem 1.1~~

$x \in \mathbb{R}$, obj = 88

subproblem 1.2

$x \in \mathbb{R}$, obj = 97

~~subproblem 1.2.1~~

$x \in \mathbb{R}$, obj = 85

~~subproblem 1.2.2~~

$x \in \mathbb{Z}$, obj = 89

Is this process guaranteed to eventually find the optimal integer solution?

Solving ILPs

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

ILP $\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$ **LP Relaxation**

1. Relax ILP

$\boldsymbol{x} \in \mathbb{R}$, **obj = 100**

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

Maximizatio

blem LPs.

UB = 90

LB = 90

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er variable.

lem LPs.

~~**subproblem**~~

$\boldsymbol{x} \in \mathbb{R}$, **obj =**

Is this process guaranteed to eventually find the optimal integer solution?

Yes, given enough constraints like $x_i \geq 3$ and $x_i \leq 3$, all variables will be bounded to their optimal integer values.

Solving ILPs

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

ILP

$\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$

1. Relax ILP

LP Relaxation

$\boldsymbol{x} \in \mathbb{R}$, obj = 100

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

Maximizatio

UB = 90
LB = 90

blem LPs.

o new LPs

bproblem on

er variable.

lem LPs.

~~subproblem~~

$\boldsymbol{x} \in \mathbb{R}$, obj =

Is this process guaranteed to eventually find the optimal integer solution?

Yes, given enough constraints like $x_i \geq 3$ and $x_i \leq 3$, all variables will be bounded to their optimal integer values.

Other techniques (cutting planes, intelligently picking which x_i to split on) can help speed up the process.