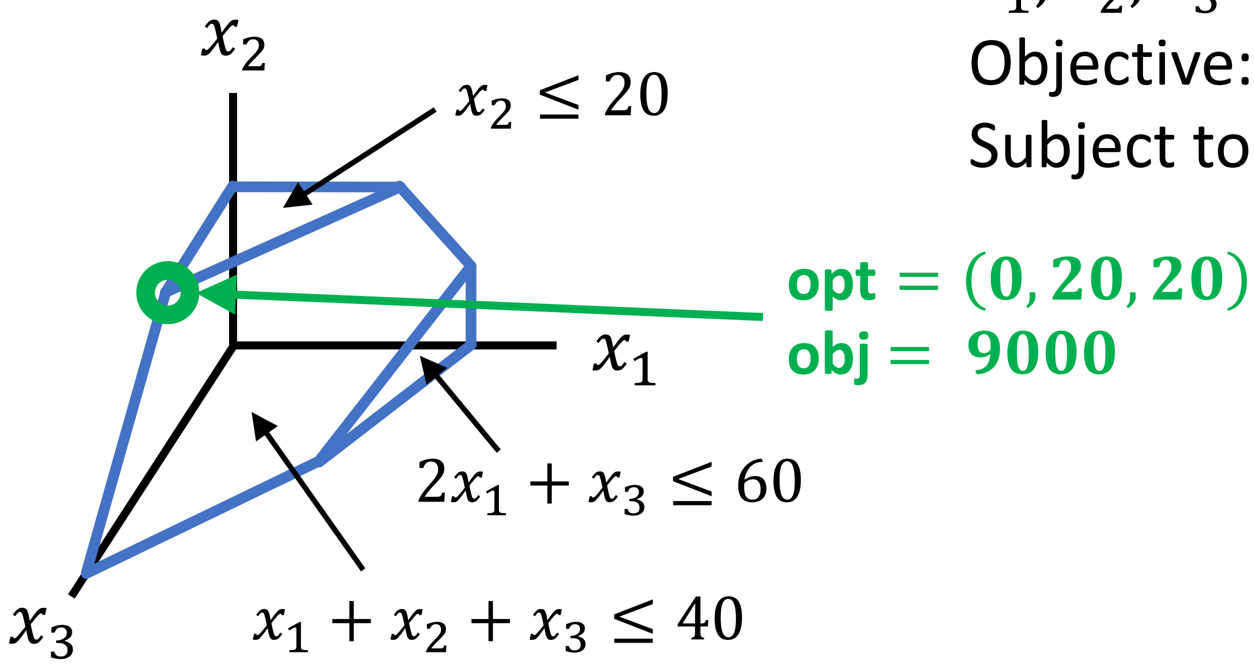
Duality
CSCI 532

Proving Optimality



$x_1, x_2, x_3 \in \mathbb{R}$

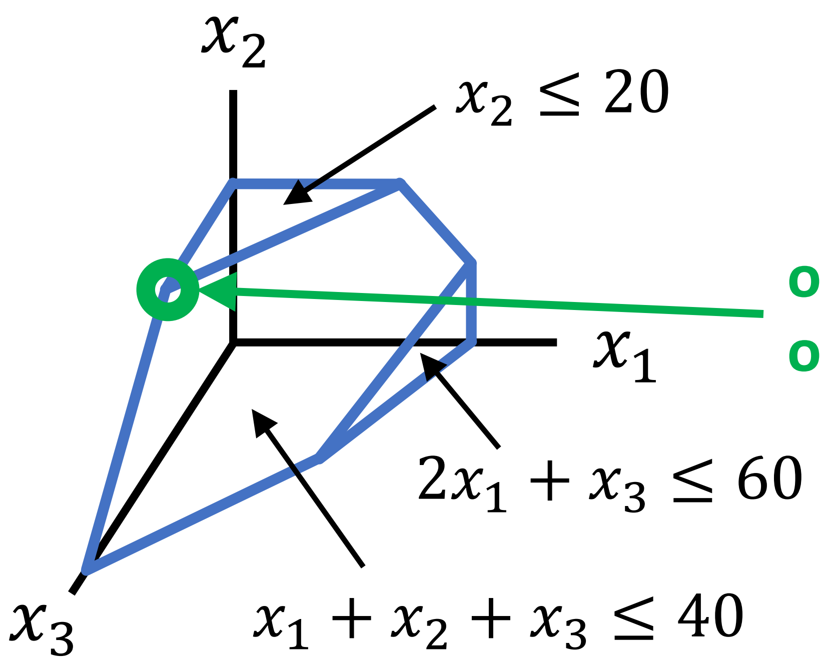
Objective: $\max 100x_1 + 300x_2 + 150x_3$

Subject to:

$x_2 \leq 20$	A
$x_1 + x_2 + x_3 \leq 40$	B
$2x_1 + x_3 \leq 60$	C
$x_1, x_2, x_3 \geq 0$	

150 (constraint A) + 150 (constraint B) $\Longrightarrow 150x_1 + 300x_2 + 150x_3 \leq 9000$

Proving Optimality



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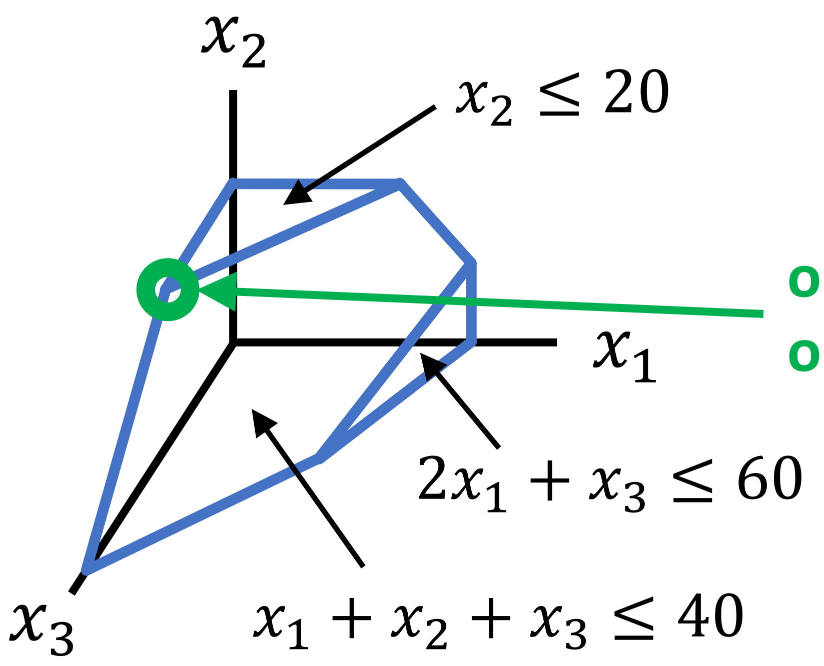
150 (constraint A) + 150 (constraint B) $\Longrightarrow 150x_1 + 300x_2 + 150x_3 \leq 9000$

$\Longrightarrow 100x_1 + 300x_2 + 150x_3 \leq 9000$

Proving Optimality



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$x_1, x_2, x_3 \geq 0$	

150 (constraint A) + 150 (constraint B) $\Longrightarrow 150x_1 + 300x_2 + 150x_3 \leq 9000$

$$\Longrightarrow 100x_1 + 300x_2 + 150x_3 \leq 9000$$

How did we get these coefficients?

Proving Optimality

Objective: $\max 100x_1 + 300x_2 + 150x_3$

Subject to:

$x_2 \leq 20$ A

$x_1 + x_2 + x_3 \leq 40$ B

$2x_1 + x_3 \leq 60$ C

$x_1, x_2 \geq 0$

Multiplier	Constraint
y_1	$x_2 \leq 20$
y_2	$x_1 + x_2 + x_3 \leq 40$
y_3	$2x_1 + x_3 \leq 60$

y_1(Constraint A) + y_2(Constraint B) + y_3(Constraint C)

Don't worry about non-negativity constraints for the moment.

Proving Optimality

Objective: $\max 100x_1 + 300x_2 + 150x_3$

Subject to:

$\boldsymbol{x_2 \leq 20}$ A

$x_1 + x_2 + x_3 \leq 40$ B

$\boldsymbol{2x_1 + x_3 \leq 60}$ C

$x_1, x_2 \geq 0$

Multiplier	Constraint
y_1	$x_2 \leq 20$
y_2	$x_1 + x_2 + x_3 \leq 40$
y_3	$2x_1 + x_3 \leq 60$

y_1(Constraint A) + y_2(Constraint B) + y_3(Constraint C)

$\Longrightarrow \boldsymbol{y_1 x_2} + y_2 x_1 + y_2 x_2 + y_2 x_3 + \boldsymbol{2y_3 x_1 + y_3 x_3} \leq \boldsymbol{20y_1} + 40y_2 + \boldsymbol{60y_3}$

Proving Optimality

Multiplier	Constraint
y_1	$x_2 \leq 20$
y_2	$x_1 + x_2 + x_3 \leq 40$
y_3	$2x_1 + x_3 \leq 60$

Objective: $\max 100x_1 + 300x_2 + 150x_3$

Subject to:
$x_2 \leq 20$ A
$x_1 + x_2 + x_3 \leq 40$ B
$2x_1 + x_3 \leq 60$ C
$x_1, x_2 \geq 0$

y_1(Constraint A) + y_2(Constraint B) + y_3(Constraint C)

$\Longrightarrow y_1x_2 + y_2x_1 + y_2x_2 + y_2x_3 + 2y_3x_1 + y_3x_3 \leq 20y_1 + 40y_2 + 60y_3$

y_i need to be ≥ 0, because multiplying by negative swaps the inequality sign.

Proving Optimality

Multiplier	Constraint
y_1	$x_2 \leq 20$
y_2	$x_1 + x_2 + x_3 \leq 40$
y_3	$2x_1 + x_3 \leq 60$

Objective: $\max 100x_1 + 300x_2 + 150x_3$

Subject to:
$x_2 \leq 20$ A
$x_1 + x_2 + x_3 \leq 40$ B
$2x_1 + x_3 \leq 60$ C
$x_1, x_2 \geq 0$

y_1(Constraint A) + y_2(Constraint B) + y_3(Constraint C)

$\Longrightarrow y_1x_2 + y_2x_1 + y_2x_2 + y_2x_3 + 2y_3x_1 + y_3x_3 \leq 20y_1 + 40y_2 + 60y_3$

$\Longrightarrow (y_2 + 2y_3)x_1 + (y_1 + y_2)x_2 + (y_2 + y_3)x_3 \leq 20y_1 + 40y_2 + 60y_3$

Proving Optimality

Objective: $\max 100x_1 + 300x_2 + 150x_3$

Subject to:

$x_2 \leq 20$ A

$x_1 + x_2 + x_3 \leq 40$ B

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$x_1, x_2 \geq 0$

Multiplier	Constraint
y_1	$x_2 \leq 20$
y_2	$x_1 + x_2 + x_3 \leq 40$
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y_1(Constraint A) + y_2(Constraint B) + y_3(Constraint C)

$$\Longrightarrow y_1x_2 + y_2x_1 + y_2x_2 + y_2x_3 + 2y_3x_1 + y_3x_3 \leq 20y_1 + 40y_2 + 60y_3$$

$$\Longrightarrow (y_2 + 2y_3)x_1 + (y_1 + y_2)x_2 + (y_2 + y_3)x_3 \leq 20y_1 + 40y_2 + 60y_3$$

Want to make **this** look like **this**.

Proving Optimality

Multiplier	Constraint
y_1	$x_2 \leq 20$
y_2	$x_1 + x_2 + x_3 \leq 40$
y_3	$2x_1 + x_3 \leq 60$

Objective: $\max 100x_1 + 300x_2 + 150x_3$

Subject to:
$x_2 \leq 20$ A
$x_1 + x_2 + x_3 \leq 40$ B
$2x_1 + x_3 \leq 60$ C
$x_1, x_2 \geq 0$

y_1(Constraint A) + y_2(Constraint B) + y_3(Constraint C)

$\Longrightarrow y_1x_2 + y_2x_1 + y_2x_2 + y_2x_3 + 2y_3x_1 + y_3x_3 \leq 20y_1 + 40y_2 + 60y_3$

$\Longrightarrow (y_2 + 2y_3)x_1 + (y_1 + y_2)x_2 + (y_2 + y_3)x_3 \leq 20y_1 + 40y_2 + 60y_3$

$\Longrightarrow 100x_1 + 300x_2 + 150x_3 \leq 20y_1 + 40y_2 + 60y_3$, If:
$y_2 + 2y_3 \geq 100$
$y_1 + y_2 \geq 300$
$y_2 + y_3 \geq 150$
$y_1, y_2, y_3 \geq 0$

Proving Optimality

Objective: $\max \mathbf{100}x_1 + 300x_2 + 150x_3$

Subject to:

$x_2 \leq 20$ A

$x_1 + x_2 + x_3 \leq 40$ B

$2x_1 + x_3 \leq 60$ C

$x_1, x_2 \geq 0$

Multiplier	Constraint
y_1	$x_2 \leq 20$
y_2	$x_1 + x_2 + x_3 \leq 40$
y_3	$2x_1 + x_3 \leq 60$

y_1(Constraint A) + y_2(Constraint B) + y_3(Constraint C)

$\Longrightarrow y_1x_2 + y_2x_1 + y_2x_2 + y_2x_3 + 2y_3x_1 + y_3x_3 \leq 20y_1 + 40y_2 + 60y_3$

$\Longrightarrow (\boldsymbol{y_2 + 2y_3})x_1 + (y_1 + y_2)x_2 + (y_2 + y_3)x_3 \leq 20y_1 + 40y_2 + 60y_3$

$\Longrightarrow 100x_1 + 300x_2 + 150x_3 \leq \mathbf{20}\boldsymbol{y_1} + \mathbf{40}\boldsymbol{y_2} + \mathbf{60}\boldsymbol{y_3}$, If: $y_2 + 2y_3 \geq 100$

$y_1 + y_2 \geq 300$

$y_2 + y_3 \geq 150$

$y_1, y_2, y_3 \geq 0$

$\geq$ because the **coefficient** needs to bound the **objective** from above so that we get a **bound on the the objective function.**

Proving Optimality

Objective: $\max 100x_1 + 300x_2 + 150x_3$

Subject to:

$x_2 \leq 20$ A

$x_1 + x_2 + x_3 \leq 40$ B

$2x_1 + x_3 \leq 60$ C

$x_1, x_2 \geq 0$

Multiplier	Constraint
y_1	$x_2 \leq 20$
y_2	$x_1 + x_2 + x_3 \leq 40$
y_3	$2x_1 + x_3 \leq 60$

y_1(Constraint A) + y_2(Constraint B) + y_3(Constraint C)

$\Longrightarrow y_1x_2 + y_2x_1 + y_2x_2 + y_2x_3 + 2y_3x_1 + y_3x_3 \leq 20y_1 + 40y_2 + 60y_3$

$\Longrightarrow (y_2 + 2y_3)x_1 + (y_1 + y_2)x_2 + (y_2 + y_3)x_3 \leq 20y_1 + 40y_2 + 60y_3$

$\Longrightarrow 100x_1 + 300x_2 + 150x_3 \leq 20y_1 + 40y_2 + 60y_3$, If: $y_2 + 2y_3 \geq 100$

$y_1 + y_2 \geq 300$

$y_2 + y_3 \geq 150$

$y_1, y_2, y_3 \geq 0$

So, we need to find y values that minimize **this**

Proving Optimality

Objective: $\max 100x_1 + 300x_2 + 150x_3$

Subject to:

$x_2 \leq 20$ A

$x_1 + x_2 + x_3 \leq 40$ B

$2x_1 + x_3 \leq 60$ C

$x_1, x_2 \geq 0$

Multiplier	Constraint
y_1	$x_2 \leq 20$
y_2	$x_1 + x_2 + x_3 \leq 40$
y_3	$2x_1 + x_3 \leq 60$

y_1(Constraint A) + y_2(Constraint B) + y_3(Constraint C)

$$\Longrightarrow y_1x_2 + y_2x_1 + y_2x_2 + y_2x_3 + 2y_3x_1 + y_3x_3 \leq 20y_1 + 40y_2 + 60y_3$$

$$\Longrightarrow (y_2 + 2y_3)x_1 + (y_1 + y_2)x_2 + (y_2 + y_3)x_3 \leq 20y_1 + 40y_2 + 60y_3$$

$$\Longrightarrow 100x_1 + 300x_2 + 150x_3 \leq 20y_1 + 40y_2 + 60y_3,$$ If:

$$y_2 + 2y_3 \geq 100$$
$$y_1 + y_2 \geq 300$$
$$y_2 + y_3 \geq 150$$
$$y_1, y_2, y_3 \geq 0$$

So, we need to find y values that minimize **this** without violating any of **these**.

Proving Optimality

Multiplier	Constraint
y_1	$x_2 \le 20$
y_2	$x_1 + x$ …

Objective: $\max 100x_1 + 300x_2 + 150x_3$

Subject to:

$x_2 \le 20$ A

$x_1 + x_2 + x_3 \le 40$ B

$2x_1 + x_3 \le 60$ C

$x_1, x_2 \ge 0$

y_1(Constraint A) …

… $40y_2 + 60y_3$

… $y_3)x_3 \le 20y_1 + 40y_2 + 60y_3$

… $x_2 + 150x_3 \le 20y_1 + 40y_2 + 60y_3$, If: $y_2 + 2y_3 \ge 100$

$y_1 + y_2 \ge 300$

$y_2 + y_3 \ge 150$

$y_1, y_2, y_3 \ge 0$

Linear Programmize™ It!

So, we need to find y values that minimize **this** without violating any of **these**.

Dual

Objective: $\max 100x_1 + 300x_2 + 150x_3$

Subject to:

$x_2 \leq 20$ A

$x_1 + x_2 + x_3 \leq 40$ B

$2x_1 + x_3 \leq 60$ C

$x_1, x_2 \geq 0$

Objective: $\min 20y_1 + 40y_2 + 60y_3$

Subject to:

$y_2 + 2y_3 \geq 100$

$y_1 + y_2 \geq 300$

$y_2 + y_3 \geq 150$

$y_1, y_2, y_3 \geq 0$

y_1(Constraint A) + y_2(Constraint B) + y_3(Constraint C)

$\Longrightarrow y_1x_2 + y_2x_1 + y_2x_2 + y_2x_3 + 2y_3x_1 + y_3x_3 \leq 20y_1 + 40y_2 + 60y_3$

$\Longrightarrow (y_2 + 2y_3)x_1 + (y_1 + y_2)x_2 + (y_2 + y_3)x_3 \leq 20y_1 + 40y_2 + 60y_3$

$\Longrightarrow 100x_1 + 300x_2 + 150x_3 \leq 20y_1 + 40y_2 + 60y_3,$ If:

$y_2 + 2y_3 \geq 100$

$y_1 + y_2 \geq 300$

$y_2 + y_3 \geq 150$

$y_1, y_2, y_3 \geq 0$

Dual

Objective: $\max 100x_1 + 300x_2 + 150x_3$
Subject to:
$x_2 \leq 20$ A
$x_1 + x_2 + x_3 \leq 40$ B
$2x_1 + x_3 \leq 60$ C
$x_1, x_2 \geq 0$

Objective: $\min 20y_1 + 40y_2 + 60y_3$
Subject to:
$y_2 + 2y_3 \geq 100$
$y_1 + y_2 \geq 300$
$y_2 + y_3 \geq 150$
$y_1, y_2, y_3 \geq 0$

y_1(Constraint A) + y_2(Constraint B) + y_3(Constraint C)

$\Rightarrow y_1x_2 + y_2x_1 + y_2x_2 + y_2x_3 + 2y_3x_1 + y_3x_3 \leq 20y_1 + 40y_2 + 60y_3$

$\Rightarrow (y_2 + 2y_3)x_1 + (y_1 + y_2)x_2 + (y_2 + y_3)x_3 \leq 20y_1 + 40y_2 + 60y_3$

$\Rightarrow 100x_1 + 300x_2 + 150x_3 \leq 20y_1 + 40y_2 + 60y_3$, If:
$y_2 + 2y_3 \geq 100$
$y_1 + y_2 \geq 300$
$y_2 + y_3 \geq 150$
$y_1, y_2, y_3 \geq 0$

Roughly, we've made an LP that turned **constraints** into **variables** and **variables** into **constraints**.

Dual

Objective:	$\max 100x_1 + 300x_2 + 150x_3$		Objective:	$\min 20y_1 + 40y_2 + 60y_3$
Subject to:	$0x_1 + 1x_2 + 0x_3 \leq 20$	A	Subject to:	$y_2 + 2y_3 \geq 100$
	$1x_1 + 1x_2 + 1x_3 \leq 40$	B		$y_1 + y_2 \geq 300$
	$2x_1 + 0x_2 + 1x_3 \leq 60$	C		$y_2 + y_3 \geq 150$
	$x_1, x_2 \geq 0$			$y_1, y_2, y_3 \geq 0$

y_1(Constraint A) + y_2(Constraint B) + y_3(Constraint C)

$\implies y_1x_2 + y_2x_1 + y_2x_2 + y_2x_3 + 2y_3x_1 + y_3x_3 \leq 20y_1 + 40y_2 + 60y_3$

$\implies (y_2 + 2y_3)x_1 + (y_1 + y_2)x_2 + (y_2 + y_3)x_3 \leq 20y_1 + 40y_2 + 60y_3$

$\implies 100x_1 + 300x_2 + 150x_3 \leq 20y_1 + 40y_2 + 60y_3$, If: $y_2 + 2y_3 \geq 100$

$y_1 + y_2 \geq 300$

$y_2 + y_3 \geq 150$

$y_1, y_2, y_3 \geq 0$

Dual

Primal

Objective: max $c^T x$

Subject to: $A x \leq b$

$x \geq 0$

Dual

Objective: min $b^T y$

Subject to: $A^T y \geq c$

$y \geq 0$

Objective: $\max \begin{bmatrix} 100 & 300 & 150 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Subject to: $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \leq \begin{bmatrix} 20 \\ 40 \\ 60 \end{bmatrix}$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Objective: $\min \begin{bmatrix} 20 & 40 & 60 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

Subject to: $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \geq \begin{bmatrix} 100 \\ 300 \\ 150 \end{bmatrix}$

$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Duality

Primal

Objective: $\max c^T x$

Subject to: $A x \leq b$

$x \geq 0$

Dual

Objective: $\min b^T y$

Subject to: $A^T y \geq c$

$y \geq 0$

Theorem: The dual of a dual is the original primal.

Proof: ?

Duality

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Objective: $\max c^T x$

Subject to: $A x \leq b$

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Objective: $\min b^T y$

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Theorem: The dual of a dual is the original primal.

Proof:

Objective: $\min b^T y$

Subject to: $A^T y \geq c$

$y \geq 0$

Duality

Primal

Objective: max $c^T x$

Subject to: $A x \leq b$

$x \geq 0$

Dual

Objective: min $b^T y$

Subject to: $A^T y \geq c$

$y \geq 0$

Theorem: The dual of a dual is the original primal.

Proof:

Objective: min $b^T y$	Objective: max $-b^T y$
Subject to: $A^T y \geq c$ $\rightarrow$	Subject to: $-A^T y \leq -c$
$y \geq 0$	$y \geq 0$

$\boxed{\text{Standard Form}}$

Duality

Primal

Objective: max $c^T x$

Subject to: $A x \leq b$

$x \geq 0$

Dual

Objective: min $b^T y$

Subject to: $A^T y \geq c$

$y \geq 0$

Theorem: The dual of a dual is the original primal.

Proof:

Objective: min $b^T y$ Subject to: $A^T y \geq c$ $y \geq 0$	Objective: max $-b^T y$ Subject to: $-A^T y \leq -c$ $y \geq 0$	Objective: min $-c^T z$ Subject to: $-{A^T}^T z \geq -b$ $z \geq 0$

$\xrightarrow{\boxed{\text{Standard Form}}}$ $\xrightarrow{\boxed{\text{Dual}}}$

Duality

Primal

Objective: $\max c^T x$

Subject to: $A x \leq b$

$x \geq 0$

Dual

Objective: $\min b^T y$

Subject to: $A^T y \geq c$

$y \geq 0$

Theorem: The dual of a dual is the original primal.

Proof:

Objective: $\min b^T y$
Subject to: $A^T y \geq c$
$y \geq 0$

$\rightarrow$

Objective: $\max -b^T y$
Subject to: $-A^T y \leq -c$
$y \geq 0$

$\rightarrow$

Objective: $\min -c^T z$
Subject to: $-{A^T}^T z \geq -b$
$z \geq 0$

$\rightarrow$

Objective: $\max c^T z$
Subject to: $A z \leq b$
$z \geq 0$

Standard Form

Dual

Standard Form

Duality

Primal

Objective: $\max c^T x$

Subject to: $A x \leq b$

$x \geq 0$

Dual

Objective: $\min b^T y$

Subject to: $A^T y \geq c$

$y \geq 0$

Theorem (Weak Duality): $c^T x^* \leq b^T y^*$, for all feasible solutions to the primal x^*, and all feasible solutions to the dual y^*.

Proof: ?

I.e., The objective value of every feasible solution to the primal is less than or equal to the objective value of every feasible solution to the dual.

Duality

Primal

Objective: max $c^T x$

Subject to: $A x \leq b$

$x \geq 0$

Dual

Objective: min $b^T y$

Subject to: $A^T y \geq c$

$y \geq 0$

Theorem (Weak Duality): $c^T x^* \leq b^T y^*$, for all feasible solutions to the primal x^*, and all feasible solutions to the dual y^*.

Proof:

$$c^T x^* \leq (A^T y^*)^T x^*$$

Since $A^T y \geq c$

Duality

Primal

Objective: $\max c^T x$

Subject to: $A x \leq b$

$x \geq 0$

Dual

Objective: $\min b^T y$

Subject to: $A^T y \geq c$

$y \geq 0$

Theorem (Weak Duality): $c^T x^* \leq b^T y^*$, for all feasible solutions to the primal x^*, and all feasible solutions to the dual y^*.

Proof:

$$c^T x^* \leq (A^T y^*)^T x^* = (y^{*T} A) x^*$$

Since transpose of multiplication is multiplication of transposes (reversed)

Duality

Primal

Objective: $\max c^T x$

Subject to: $A x \leq b$

$x \geq 0$

Dual

Objective: $\min b^T y$

Subject to: $A^T y \geq c$

$y \geq 0$

Theorem (Weak Duality): $c^T x^* \leq b^T y^*$, for all feasible solutions to the primal x^*, and all feasible solutions to the dual y^*.

Proof:

$$c^T x^* \leq (A^T y^*)^T x^* = (y^{*T} A) x^* = y^{*T} (A x^*)$$

Since matrix multiplication is associative

Duality

Primal

Objective: $\max c^T x$

Subject to: $A x \leq b$

$x \geq 0$

Dual

Objective: $\min b^T y$

Subject to: $A^T y \geq c$

$y \geq 0$

Theorem (Weak Duality): $c^T x^* \leq b^T y^*$, for all feasible solutions to the primal x^*, and all feasible solutions to the dual y^*.

Proof:

$$c^T x^* \leq (A^T y^*)^T x^* = (y^{*T} A) x^* = y^{*T} (A x^*) \leq y^{*T} b$$

Since $A x \leq b$

Duality

Primal

Objective: max $c^T x$

Subject to: $A x \leq b$

$x \geq 0$

Dual

Objective: min $b^T y$

Subject to: $A^T y \geq c$

$y \geq 0$

Theorem (Weak Duality): $c^T x^* \leq b^T y^*$, for all feasible solutions to the primal x^*, and all feasible solutions to the dual y^*.

Proof:

$$c^T x^* \leq (A^T y^*)^T x^* = (y^{*T} A) x^* = y^{*T} (A x^*) \leq y^{*T} b = b^T y^*$$

Since b and $\overline{y}$ are 1-dimensional vectors.

$$\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 20 \\ 40 \\ 60 \end{bmatrix} = \begin{bmatrix} 20 & 40 & 60 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Duality

Primal

Objective: $\max c^T x$

Subject to: $A x \leq b$

$x \geq 0$

Dual

Objective: $\min b^T y$

Subject to: $A^T y \geq c$

$y \geq 0$

Theorem (Weak Duality): $c^T x^* \leq b^T y^*$, for all feasible solutions to the primal x^*, and all feasible solutions to the dual y^*.

Weak Duality holds for integer linear programs and is a useful tool for approximating solutions to hard problems

Duality

Primal

Objective: $\max c^T x$

Subject to: $A x \leq b$

$x \geq 0$

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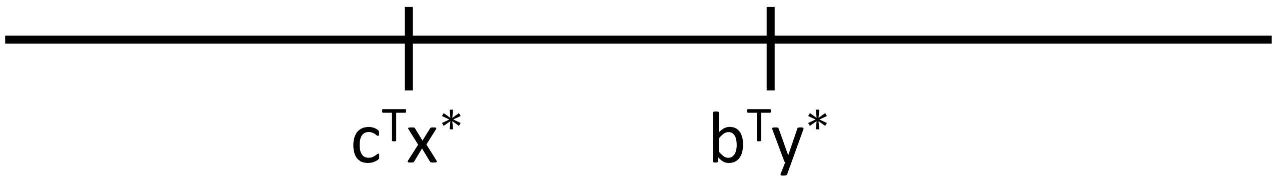
Objective: $\min b^T y$

Subject to: $A^T y \geq c$

$y \geq 0$

Theorem (Weak Duality): $c^T x^* \leq b^T y^*$, for all feasible solutions to the primal x^*, and all feasible solutions to the dual y^*.

Weak Duality holds for integer linear programs and is a useful tool for approximating solutions to hard problems



Duality

Primal

Objective: max $c^T x$

Subject to: $A x \leq b$

$x \geq 0$

Dual

Objective: min $b^T y$

Subject to: $A^T y \geq c$

$y \geq 0$

Theorem (Weak Duality): $c^T x^* \leq b^T y^*$, for all feasible solutions to the primal x^*, and all feasible solutions to the dual y^*.

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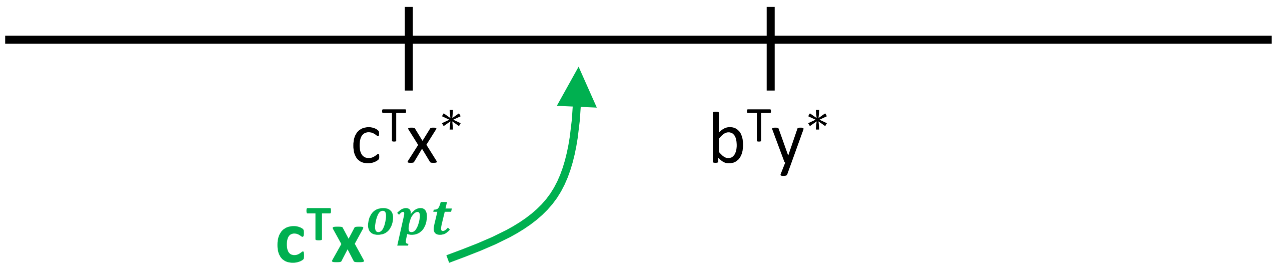

Duality

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Objective: max $c^T x$

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Objective: min $b^T y$

Subject to: $A^T y \geq c$

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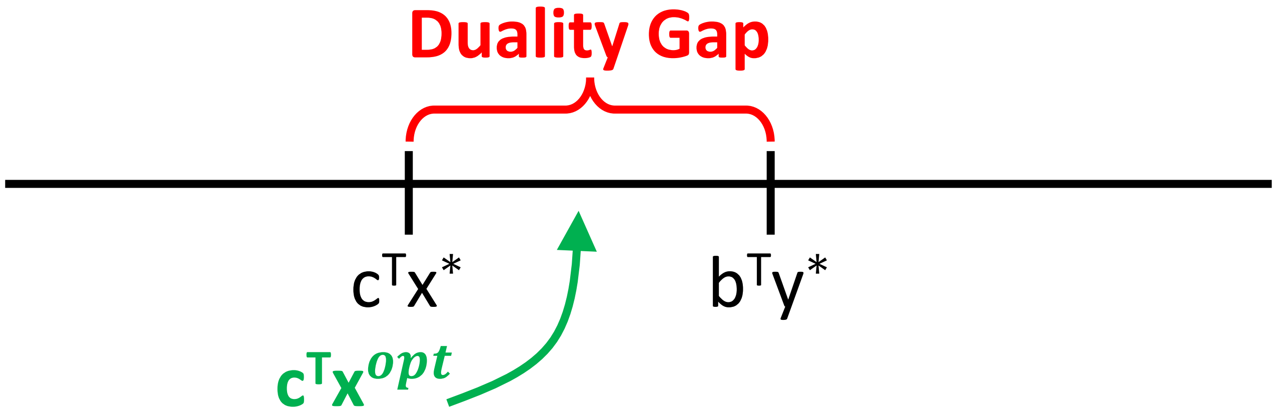

Duality

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Objective: max $c^T x$

Subject to: $A x \leq b$

$x \geq 0$

Dual

Objective: min $b^T y$

Subject to: $A^T y \geq c$

$y \geq 0$

Theorem (Strong Duality): If either the primal or the dual has a finite optimal solution, then the other does as well, and their optimal objectives are equal.

Proof: A little more complicated…

Duality

Primal

Objective: max $c^T x$

Subject to: $A x \leq b$

$x \geq 0$

Dual

Objective: min $b^T y$

Subject to: $A^T y \geq c$

$y \geq 0$

Theorem (Strong Duality): If either the primal or the dual has a finite optimal solution, then the other does as well, and their optimal objectives are equal.

Consequences:

- The optimal objective value for the primal (dual) gives you the optimal value for the dual (primal).

Duality

Primal

Objective: $\max c^T x$

Subject to: $A x \leq b$

$x \geq 0$

Dual

Objective: $\min b^T y$

Subject to: $A^T y \geq c$

$y \geq 0$

Theorem (Strong Duality): If either the primal or the dual has a finite optimal solution, then the other does as well, and their optimal objectives are equal.

Consequences:

- The optimal objective value for the primal (dual) gives you the optimal value for the dual (primal).
- If one is unbounded (i.e., infinite optimal solution) the other is infeasible (i.e., no optimal solutions).

Duality

Primal

Objective: max $c^T x$

Subject to: $A x \leq b$

$x \geq 0$

Dual

Objective: min $b^T y$

Subject to: $A^T y \geq c$

$y \geq 0$

Theorem (Strong Duality): If either the primal or the dual has a finite optimal solution, then the other does as well, and their optimal objectives are equal.

Consequences:

- The optimal objective value for the primal (dual) gives you the optimal value for the dual (primal).
- If one is unbounded (i.e., infinite optimal solution) the other is infeasible (i.e., no optimal solutions).
- Does not hold for integer linear programming.

Example

Maximum Flow

$x_{(u,v)}$ = Amount of flow on edge (u, v)

Objective: $\max \sum_{(s,v)\in E} x_{(s,v)}$

Subject to:

$$x_{(u,v)} \le c_{(u,v)}, \forall (u, v) \in E$$

$$\sum_{(w,u)\in E} x_{(w,u)} - \sum_{(u,x)\in E} x_{(u,x)} = 0,$$
$$\forall u \in V \setminus \{s, t\}$$

$$x_{(u,v)} \ge 0, \forall (u, v) \in E$$

Example

Maximum Flow

$x_{(u,v)}$ = Amount of flow on edge (u, v)

Objective: $\max \sum_{(s,v)\in E} x_{(s,v)}$

Subject to:

$$x_{(u,v)} \leq c_{(u,v)}, \forall (u, v) \in E$$

$$\sum_{(w,u)\in E} x_{(w,u)} - \sum_{(u,x)\in E} x_{(u,x)} = 0, \quad \forall u \in V \setminus \{s, t\}$$

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Shortest Path

??

Example

Maximum Flow

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$$x_{(u,v)} \geq 0, \forall (u, v) \in E$$

Shortest Path

$x_{(u,v)}$ = Amount of flow on edge (u, v)

Objective: $\max \sum_{(s,v)\in E} x_{(s,v)}$

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Shortest Path

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Example

Maximum Flow

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Subject to:

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Shortest Path

$x_{(u,v)}$ = Amount of flow on edge (u, v)

Objective: $\max \sum_{(s,v)\in E} x_{(s,v)}$

Subject to:

$$\cancel{x_{(u,v)} \leq c_{(u,v)}, \forall (u, v) \in E}$$
$$\sum_{(w,u)\in E} x_{(w,u)} - \sum_{(u,x)\in E} x_{(u,x)} = 0, \quad \forall u \in V \setminus \{s, t\}$$
$$\sum_{(s,v)\in E} x_{(s,v)} = 1$$
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Example

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Shortest Path

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Objective: $\max \sum_{(s,v)\in E} x_{(s,v)}$

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$$x_{(u,v)} \geq 0, \forall (u,v) \in E$$

Shortest Path

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Objective: $\min \sum_{(u,v)\in E} c_{(u,v)} x_{(u,v)}$

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Shortest Path

$x_{(u,v)}$ = Amount of flow on edge (u, v)

Objective: $\min \sum_{(u,v) \in E} c_{(u,v)} x_{(u,v)}$

Subject to:

$$\sum_{(w,u) \in E} x_{(w,u)} - \sum_{(u,x) \in E} x_{(u,x)} = 0, \quad \forall u \in V \setminus \{s, t\}$$

$$\sum_{(s,v) \in E} x_{(s,v)} = 1$$

$$x_{(u,v)} \geq 0, \forall (u, v) \in E$$

Shortest Path

Primal

Objective: $\min c^T x$

Subject to: $A x = b$

$x \geq 0$

Dual

Objective: $\max b^T y$

Subject to: $A^T y \leq c$

This can be derived from standard primal/dual definitions.

$x_{(u,v)}$ = Amount of flow on edge (u, v)

Objective: $\min \sum_{(u,v)\in E} c_{(u,v)} x_{(u,v)}$

Subject to:

$$\sum_{(w,u)\in E} x_{(w,u)} - \sum_{(u,x)\in E} x_{(u,x)} = 0, \quad \forall u \in V \setminus \{s, t\}$$

$$\sum_{(s,v)\in E} x_{(s,v)} = 1$$

$$x_{(u,v)} \geq 0, \forall (u, v) \in E$$

Shortest Path

Primal

Objective: $\min c^T x$

Subject to: $A x = b$

$x \geq 0$

Dual

Objective: $\max b^T y$

Subject to: $A^T y \leq c$

$x_{(u,v)}$ = Amount of flow on edge (u, v)

Objective: $\min \sum_{(u,v)\in E} c_{(u,v)} x_{(u,v)}$

Subject to:

$$\sum_{(w,u)\in E} x_{(w,u)} - \sum_{(u,x)\in E} x_{(u,x)} = 0, \quad \forall u \in V \setminus \{s, t\}$$

$$\sum_{(s,v)\in E} x_{(s,v)} = 1$$

$$x_{(u,v)} \geq 0, \forall (u, v) \in E$$

Dual Variables?

Shortest Path

Primal

Objective: $\min c^T x$

Subject to: $A x = b$

$x \geq 0$

Dual

Objective: $\max b^T y$

Subject to: $A^T y \leq c$

$x_{(u,v)}$ = Amount of flow on edge (u, v)

Objective: $\min \sum_{(u,v)\in E} c_{(u,v)} x_{(u,v)}$

Subject to:

$$\sum_{(w,u)\in E} x_{(w,u)} - \sum_{(u,x)\in E} x_{(u,x)} = 0, \quad \forall u \in V \setminus \{s,t\}$$

$$\sum_{(s,v)\in E} x_{(s,v)} = 1$$

$$x_{(u,v)} \geq 0, \forall (u, v) \in E$$

y_v = Conservation constraints (not s or t)

z = Source outflow constraint

Dual Variables? Each constraint (other than non-negativity) from the primal corresponds to a variable in the dual.

Shortest Path

Primal

Objective: $\min c^T x$

Subject to: $A x = b$

$x \geq 0$

$x_{(u,v)}$ = Amount of flow on edge (u, v)

Objective: $\min \sum_{(u,v)\in E} c_{(u,v)} x_{(u,v)}$

Subject to:

$$\sum_{(w,u)\in E} x_{(w,u)} - \sum_{(u,x)\in E} x_{(u,x)} = 0, \quad \forall u \in V \setminus \{s, t\}$$

$$\sum_{(s,v)\in E} x_{(s,v)} = 1$$

$$x_{(u,v)} \geq 0, \forall (u, v) \in E$$

Dual

Objective: $\max b^T y$

Subject to: $A^T y \leq c$

y_v = Conservation constraints (not s or t)

z = Source outflow constraint

Objective:

Objective?

Shortest Path

Primal

Objective: min $c^T x$
Subject to: $A x = b$
$x \geq 0$

Dual

Objective: max $b^T y$
Subject to: $A^T y \leq c$

$x_{(u,v)}$ = Amount of flow on edge (u, v)

Objective: $\min \sum_{(u,v)\in E} c_{(u,v)} x_{(u,v)}$

Subject to:

$$\sum_{(w,u)\in E} x_{(w,u)} - \sum_{(u,x)\in E} x_{(u,x)} = 0, \quad \forall u \in V \setminus \{s, t\}$$

$$\sum_{(s,v)\in E} x_{(s,v)} = 1$$

$$x_{(u,v)} \geq 0, \forall (u, v) \in E$$

y_v = Conservation constraints (not s or t)
z = Source outflow constraint

Objective: $\max z$

Objective? The only non-zero b values are for the source outflow constraint.

Shortest Path

Primal

Objective: min $c^T x$

Subject to: $A\,x = b$

$x \geq 0$

Dual

Objective: max $b^T y$

Subject to: $A^T y \leq c$

$x_{(u,v)}$ = Amount of flow on edge (u, v)

Objective: $\min \sum_{(u,v)\in E} c_{(u,v)} x_{(u,v)}$

Subject to:

$$\sum_{(w,u)\in E} x_{(w,u)} - \sum_{(u,x)\in E} x_{(u,x)} = 0, \quad \forall u \in V \setminus \{s, t\}$$

$$\sum_{(s,v)\in E} x_{(s,v)} = 1$$

$$x_{(u,v)} \geq 0, \forall (u, v) \in E$$

y_v = Conservation constraints (not s or t)

z = Source outflow constraint

Objective: $\max z$

Subject to:

Constraints? Turn columns of A into rows of A^T

Shortest Path

Primal

Objective: min $c^T x$

Subject to: $A\,x = b$

$x \geq 0$

$x_{(u,v)}$ = Amount of flow on edge (u,v)

Objective: $\min \sum_{(u,v)\in E} c_{(u,v)} x_{(u,v)}$

Subject to:

$$\sum_{(w,u)\in E} x_{(w,u)} - \sum_{(u,x)\in E} x_{(u,x)} = 0, \quad \forall u \in V \setminus \{s,t\}$$

$$\sum_{(s,v)\in E} x_{(s,v)} = 1$$

$$x_{(u,v)} \geq 0, \forall (u,v) \in E$$

Dual

Objective: max $b^T y$

Subject to: $A^T y \leq c$

y_v = Conservation constraints (not s or t)

z = Source outflow constraint

Objective: $\max z$

Subject to:

When does $x_{(u,v)}$ show up in A x?

Shortest Path

Primal

Objective: min $c^T x$

Subject to: $A x = b$

$x \geq 0$

$x_{(u,v)}$ = Amount of flow on edge (u, v)

Objective: $\min \sum_{(u,v)\in E} c_{(u,v)} x_{(u,v)}$

Subject to:

$\sum_{(w,u)\in E} x_{(w,u)} - \sum_{(u,x)\in E} x_{(u,x)} = 0, \quad \forall u \in V \setminus \{s,t\}$

$\sum_{(s,v)\in E} x_{(s,v)} = 1$

$x_{(u,v)} \geq 0, \forall (u, v) \in E$

Dual

Objective: max $b^T y$

Subject to: $A^T y \leq c$

y_v = Conservation constraints (not s or t)

z = Source outflow constraint

Objective: $\max z$

Subject to:

When does $x_{(u,v)}$ show up in A x?

+1 coefficient for y_v

Shortest Path

Primal

Objective: min $c^T x$

Subject to: $A\,x = b$

$x \geq 0$

Dual

Objective: max $b^T y$

Subject to: $A^T y \leq c$

$x_{(u,v)}$ = Amount of flow on edge (u, v)

Objective: $\min \sum_{(u,v)\in E} c_{(u,v)} x_{(u,v)}$

Subject to:

$$\sum_{(w,u)\in E} x_{(w,u)} - \sum_{(u,x)\in E} x_{(u,x)} = 0, \quad \forall u \in V \setminus \{s,t\}$$

$$\sum_{(s,v)\in E} x_{(s,v)} = 1$$

$$x_{(u,v)} \geq 0, \forall (u,v) \in E$$

y_v = Conservation constraints (not s or t)

z = Source outflow constraint

Objective: $\max z$

Subject to:

When does $x_{(u,v)}$ show up in A x?

+1 coefficient for y_v

-1 coefficient for y_u

Shortest Path

Primal

Objective: min $c^T x$

Subject to: $A\,x = b$

$x \geq 0$

$x_{(u,v)}$ = Amount of flow on edge (u, v)

Objective: $\min \sum_{(u,v)\in E} c_{(u,v)} x_{(u,v)}$

Subject to:

$\sum_{(w,u)\in E} x_{(w,u)} - \sum_{(u,x)\in E} x_{(u,x)} = 0, \quad \forall u \in V \setminus \{s, t\}$

$\sum_{(s,v)\in E} x_{(s,v)} = 1$

$x_{(u,v)} \geq 0, \forall (u, v) \in E$

Dual

Objective: max $b^T y$

Subject to: $A^T y \leq c$

y_v = Conservation constraints (not s or t)

z = Source outflow constraint

Objective: $\max z$

Subject to:

When does $x_{(u,v)}$ show up in A x?

+1 coefficient for y_v

-1 coefficient for y_u

+1 coefficient if $u = s$.

Shortest Path

When does $x_{(u,v)}$ show up in A x?

+1 coefficient for y_v

-1 coefficient for y_u

+1 coefficient if $u = s$.

Primal

Objective: min $c^T x$

Subject to: $A x = b$

$x \geq 0$

Dual

Objective: max $b^T y$

Subject to: $A^T y \leq c$

$x_{(u,v)}$ = Amount of flow on edge (u, v)

Objective: $\min \sum_{(u,v)\in E} c_{(u,v)} x_{(u,v)}$

Subject to:

$$\sum_{(w,u)\in E} x_{(w,u)} - \sum_{(u,x)\in E} x_{(u,x)} = 0, \quad \forall u \in V \setminus \{s,t\}$$

$$\sum_{(s,v)\in E} x_{(s,v)} = 1$$

$$x_{(u,v)} \geq 0, \forall (u,v) \in E$$

y_v = Conservation constraints (not s or t)

z = Source outflow constraint

Objective: $\max z$

Subject to:

$$y_v - y_s + z \leq c_{(s,v)}, \forall (s,v) \in E$$

Shortest Path

When does $x_{(u,v)}$ show up in A x?
+1 coefficient for y_v
-1 coefficient for y_u
+1 coefficient if $u = s$.

Primal

Objective: min $c^T x$
Subject to: $A x = b$
$x \geq 0$

$x_{(u,v)}$ = Amount of flow on edge (u, v)

Objective: $\min \sum_{(u,v)\in E} c_{(u,v)} x_{(u,v)}$

Subject to:

$\sum_{(w,u)\in E} x_{(w,u)} - \sum_{(u,x)\in E} x_{(u,x)} = 0, \quad \forall u \in V \setminus \{s, t\}$

$\sum_{(s,v)\in E} x_{(s,v)} = 1$

$x_{(u,v)} \geq 0, \forall (u, v) \in E$

Dual

Objective: max $b^T y$
Subject to: $A^T y \leq c$

y_v = Conservation constraints (not s or t)
z = Source outflow constraint

Objective: $\max z$

Subject to:

$y_v - y_s + z \leq c_{(s,v)}, \forall (s, v) \in E$

Shortest Path

When does $x_{(u,v)}$ show up in A x?
+1 coefficient for y_v
-1 coefficient for y_u
+1 coefficient if $u = s$.

Primal

Objective: min $c^T x$
Subject to: $A x = b$
$x \geq 0$

Dual

Objective: max $b^T y$
Subject to: $A^T y \leq c$

$x_{(u,v)}$ = Amount of flow on edge (u, v)

Objective: $\min \sum_{(u,v)\in E} c_{(u,v)} x_{(u,v)}$

Subject to:

$$\sum_{(w,u)\in E} x_{(w,u)} - \sum_{(u,x)\in E} x_{(u,x)} = 0, \quad \forall u \in V \setminus \{s, t\}$$

$$\sum_{(s,v)\in E} x_{(s,v)} = 1$$

$$x_{(u,v)} \geq 0, \forall (u, v) \in E$$

y_v = Conservation constraints ~~(not s or t)~~
z = Source outflow constraint

Objective: $\max z$

Subject to:

$$y_v - y_s + z \leq c_{(s,v)}, \forall (s, v) \in E$$

We can extend the representation of flow out/in flow of a node to s and t.

Shortest Path

When does $x_{(u,v)}$ show up in A x?
+1 coefficient for y_v
-1 coefficient for y_u
+1 coefficient if $u = s$.

Primal

Objective: min $c^T x$
Subject to: $A x = b$
$x \geq 0$

$x_{(u,v)}$ = Amount of flow on edge (u, v)

Objective: $\min \sum_{(u,v)\in E} c_{(u,v)} x_{(u,v)}$

Subject to:

$$\sum_{(w,u)\in E} x_{(w,u)} - \sum_{(u,x)\in E} x_{(u,x)} = 0, \quad \forall u \in V \setminus \{s, t\}$$

$$\sum_{(s,v)\in E} x_{(s,v)} = 1$$

$$x_{(u,v)} \geq 0, \forall (u, v) \in E$$

Dual

Objective: max $b^T y$
Subject to: $A^T y \leq c$

y_v = Conservation constraints
z = Source outflow constraint

Objective: $\max z$

Subject to:

$$y_v - y_s + z \leq c_{(s,v)}, \forall (s, v) \in E$$

$$y_v - y_u \leq c_{(u,v)}, \forall (u, v) \in E, u \neq s$$

Shortest Path

When does $x_{(u,v)}$ show up in A x?
+1 coefficient for y_v
-1 coefficient for y_u
+1 coefficient if $u = s$.

Primal

Objective: min $c^T x$
Subject to: $A x = b$
$x \geq 0$

$x_{(u,v)}$ = Amount of flow on edge (u, v)

Objective: $\min \sum_{(u,v)\in E} c_{(u,v)} x_{(u,v)}$

Subject to:

$\sum_{(w,u)\in E} x_{(w,u)} - \sum_{(u,x)\in E} x_{(u,x)} = 0, \quad \forall u \in V \setminus \{s,t\}$

$\sum_{(s,v)\in E} x_{(s,v)} = 1$

$x_{(u,v)} \geq 0, \forall (u,v) \in E$

Vertex Property

Dual

Objective: max $b^T y$
Subject to: $A^T y \leq c$

y_v = Conservation constraints
z = Source outflow constraint

Objective: $\max z$

Subject to:

Edge Property

$y_v - y_s + z \leq c_{(s,v)}, \forall (s,v) \in E$

$y_v - y_u \leq c_{(u,v)}, \forall (u,v) \in E, u \neq s$

Shortest Path

Primal

Objective: min $c^T x$

Subject to: $A x = b$

$x \geq 0$

Dual

Objective: max $b^T y$

Subject to: $A^T y \leq c$

$x_{(u,v)}$ = Amount of flow on edge (u, v)

Objective: $\min \sum_{(u,v)\in E} c_{(u,v)} x_{(u,v)}$

Subject to:

$$\sum_{(w,u)\in E} x_{(w,u)} - \sum_{(u,x)\in E} x_{(u,x)} = 0,$$
$$\forall u \in V \setminus \{s, t\}$$
$$\sum_{(s,v)\in E} x_{(s,v)} = 1$$
$$x_{(u,v)} \geq 0, \forall (u, v) \in E$$

y_v = Conservation constraints

z = Source outflow constraint

Objective: $\max y_t - y_s$

Subject to:

$$y_v - y_s + z \leq c_{(s,v)}, \forall (s, v) \in E$$
$$y_v - y_u \leq c_{(u,v)}, \forall (u, v) \in E, u \neq s$$

Shortest Path

Primal

Objective: min $c^T x$
Subject to: $A x = b$
$x \geq 0$

$x_{(u,v)}$ = Amount of flow on edge (u, v)

Objective: $\min \sum_{(u,v)\in E} c_{(u,v)} x_{(u,v)}$

Subject to:

$$\sum_{(w,u)\in E} x_{(w,u)} - \sum_{(u,x)\in E} x_{(u,x)} = 0, \quad \forall u \in V \setminus \{s, t\}$$

$$\sum_{(s,v)\in E} x_{(s,v)} = 1$$

$$x_{(u,v)} \geq 0, \forall (u, v) \in E$$

Dual

Objective: max $b^T y$
Subject to: $A^T y \leq c$

y_v = Conservation constraints
~~z = Source outflow constraint~~

Objective: $\max y_t - y_s$

Subject to:

$$\cancel{y_v - y_s + z \leq c_{(s,v)}, \forall (s, v) \in E}$$

$$y_v - y_u \leq c_{(u,v)}, \forall (u, v) \in E$$

Shortest Path

Primal

Objective: $\min c^T x$

Subject to: $A x = b$

$x \geq 0$

Dual

Objective: $\max b^T y$

Subject to: $A^T y \leq c$

$x_{(u,v)}$ = Amount of flow on edge (u, v)

Objective: $\min \sum_{(u,v)\in E} c_{(u,v)} x_{(u,v)}$

Subject to:

$$\sum_{(w,u)\in E} x_{(w,u)} - \sum_{(u,x)\in E} x_{(u,x)} = 0, \quad \forall u \in V \setminus \{s, t\}$$

$$\sum_{(s,v)\in E} x_{(s,v)} = 1$$

$$x_{(u,v)} \geq 0, \forall (u, v) \in E$$

y_v = Conservation constraints

Objective: $\max y_t - y_s$

Subject to:

$$y_v - y_u \leq c_{(u,v)}, \forall (u, v) \in E$$

The rise in value between neighboring nodes can't exceed the cost of the node.

Shortest Path

Primal

Objective: min $c^T x$

Subject to: $A x = b$

$x \geq 0$

$x_{(u,v)}$ = Amount of flow on edge (u, v)

Objective: $\min \sum_{(u,v)\in E} c_{(u,v)} x_{(u,v)}$

Subject to:

$$\sum_{(w,u)\in E} x_{(w,u)} - \sum_{(u,x)\in E} x_{(u,x)} = 0, \quad \forall u \in V \setminus \{s,t\}$$

$$\sum_{(s,v)\in E} x_{(s,v)} = 1$$

$$x_{(u,v)} \geq 0, \forall (u,v) \in E$$

Dual

Objective: max $b^T y$

Subject to: $A^T y \leq c$

y_v = Roughly, the distance from v to s.

Objective: $\max y_t - y_s$

Subject to:

$$y_v - y_u \leq c_{(u,v)}, \forall (u,v) \in E$$

The rise in value between neighboring nodes can't exceed the cost of the node.

Shortest Path

Primal

Objective: $\min c^T x$

Subject to: $A x = b$

$x \geq 0$

Dual

Objective: $\max b^T y$

Subject to: $A^T y \leq c$

$x_{(u,v)}$ = Amount of flow on edge (u, v)

Objective: $\min \sum_{(u,v)\in E} c_{(u,v)} x_{(u,v)}$

Subject to:

$$\sum_{(w,u)\in E} x_{(w,u)} - \sum_{(u,x)\in E} x_{(u,x)} = 0, \quad \forall u \in V \setminus \{s,t\}$$

$$\sum_{(s,v)\in E} x_{(s,v)} = 1$$

$$x_{(u,v)} \geq 0, \forall (u,v) \in E$$

y_v = Roughly, the distance from v to s.

Objective: $\max y_t - y_s$

Subject to:

$$y_v - y_u \leq c_{(u,v)}, \forall (u,v) \in E$$

Interpretation?

Shortest Path

Primal

Objective: $\min c^T x$

Subject to: $A x = b$

$x \geq 0$

$x_{(u,v)}$ = Amount of flow on edge (u, v)

Objective: $\min \sum_{(u,v)\in E} c_{(u,v)} x_{(u,v)}$

Subject to:

$$\sum_{(w,u)\in E} x_{(w,u)} - \sum_{(u,x)\in E} x_{(u,x)} = 0, \quad \forall u \in V \setminus \{s, t\}$$

$$\sum_{(s,v)\in E} x_{(s,v)} = 1$$

$$x_{(u,v)} \geq 0, \forall (u, v) \in E$$

Dual

Objective: $\max b^T y$

Subject to: $A^T y \leq c$

y_v = Roughly, the distance from v to s.

Objective: $\max y_t - y_s$

Subject to:

$$y_v - y_u \leq c_{(u,v)}, \forall (u, v) \in E, u \neq s$$

Interpretation? How far apart can I pull s and t without exceeding edge costs.