Linear Programming
CSCI 532

Max Flow

Decision Variables:

- Real numbers = solvable in polynomial time (called LP).
- Integers = not (yet?) solvable in polynomial time (called integer linear program – ILP).

$\overbrace{x_e = \text{Amount of flow on edge } e.}$

Objective:

- Can be minimization or maximization.
- Must be linear combinations of variables x_i (e.g. $a_1 x_1 + \cdots + a_n x_n$ for constants a_i, not $a_i x_1 x_2$).

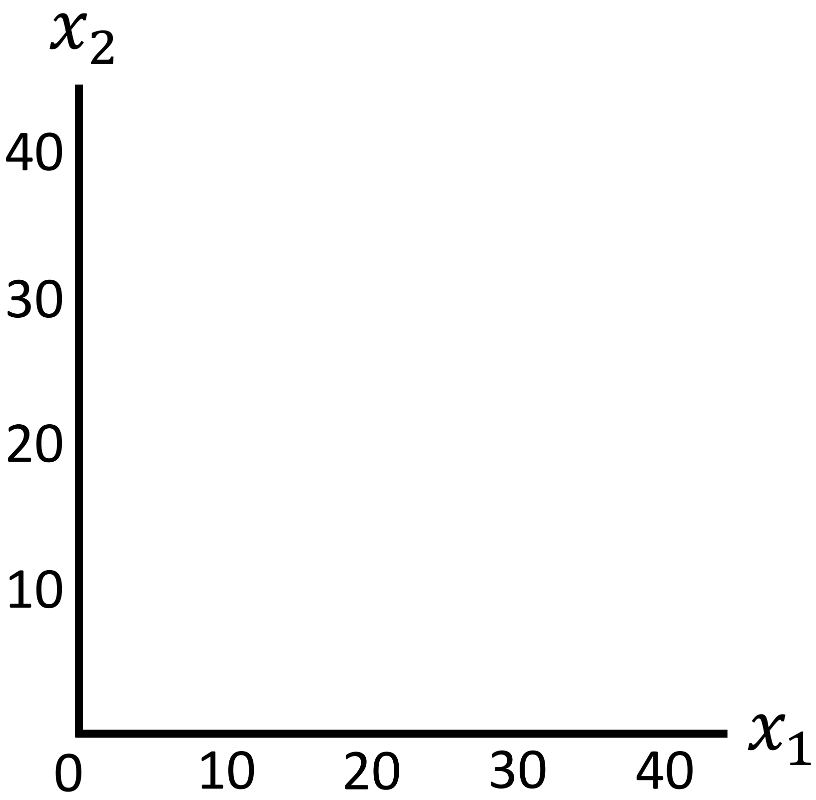
Objective: $\max \sum_{e \in \text{out}(s)} x_e$

Subject to:
$$\begin{cases} x_e \leq \text{capacity}_e, \forall e \in E \\ \sum_{e \in \text{in}(v)} x_e - \sum_{e \in \text{out}(v)} x_e = 0, \forall v \in V \setminus \{s, t\} \\ x_e \geq 0, \forall e \in E \end{cases}$$

Constraints:

- Can be $\leq$, $\geq$, $=$.
- Must be linear combinations of variables.

Example

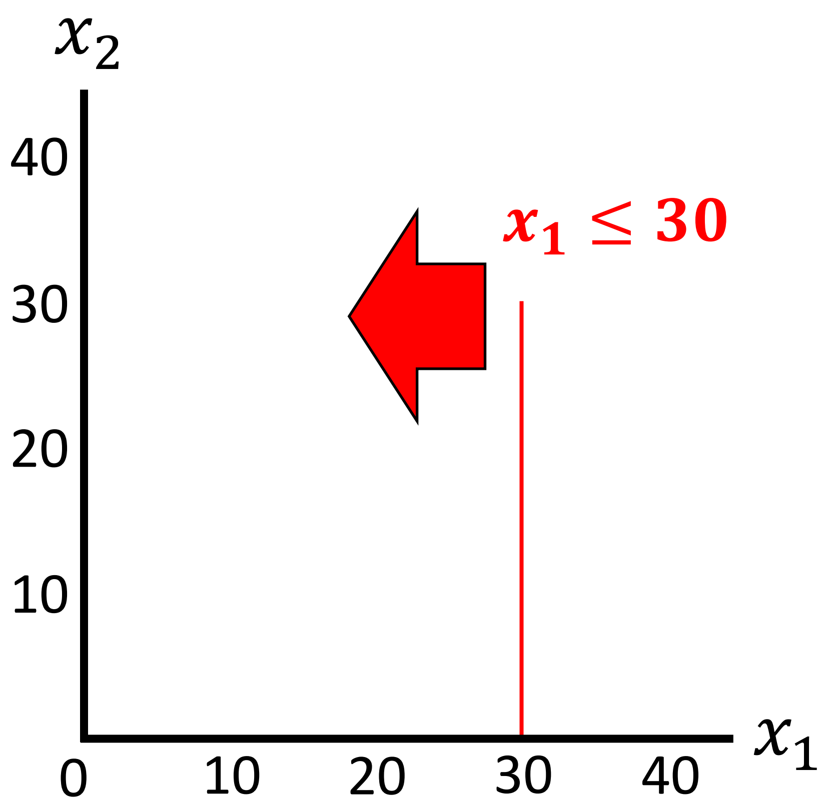


$x_1, x_2 \in \mathbb{R}$

Objective: $\max 100x_1 + 300x_2$

Subject to:
$$x_1 \leq 30$$
$$x_2 \leq 20$$
$$x_1 + x_2 \leq 40$$
$$x_1, x_2 \geq 0$$

Example



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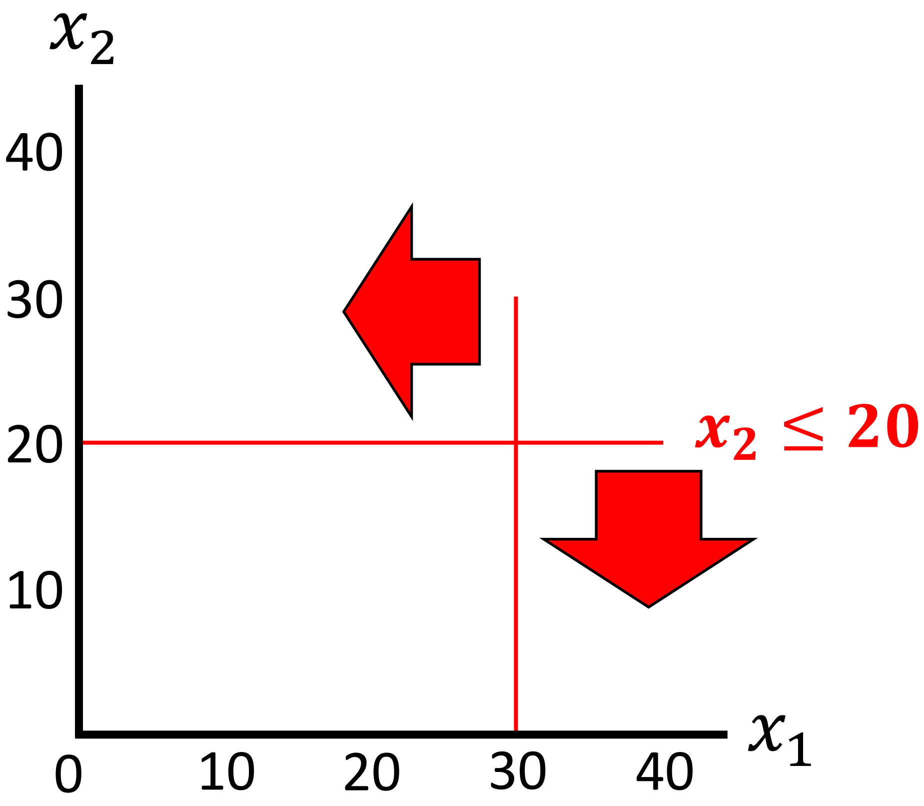

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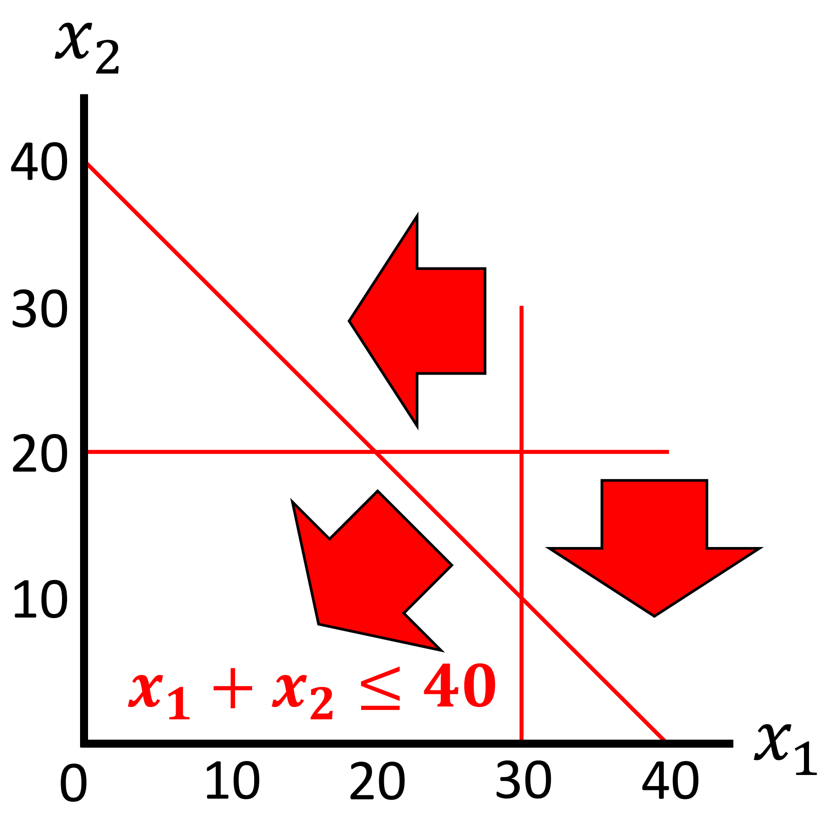

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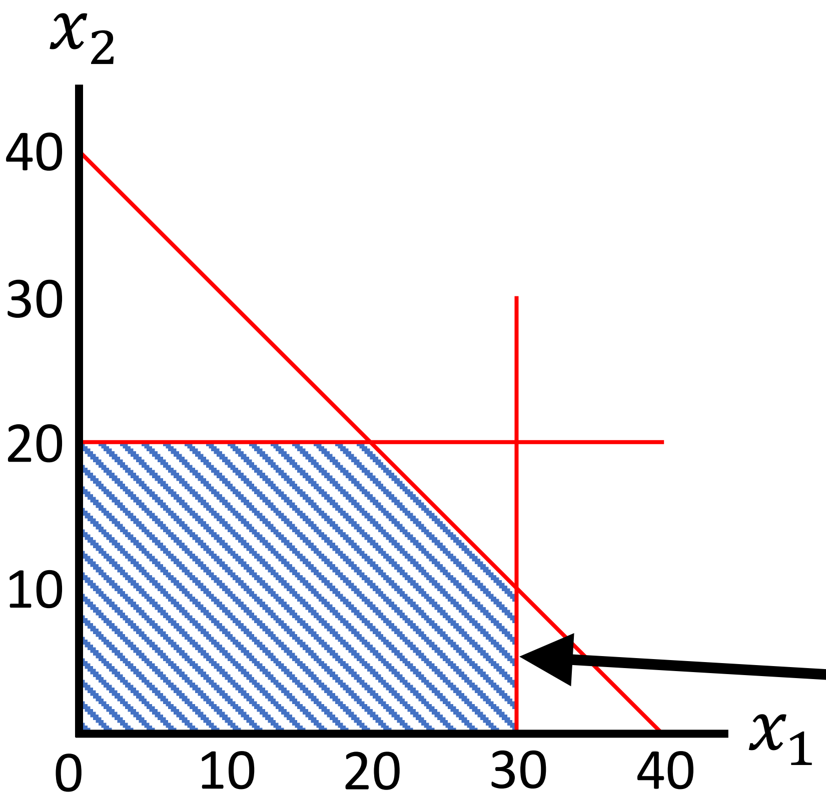
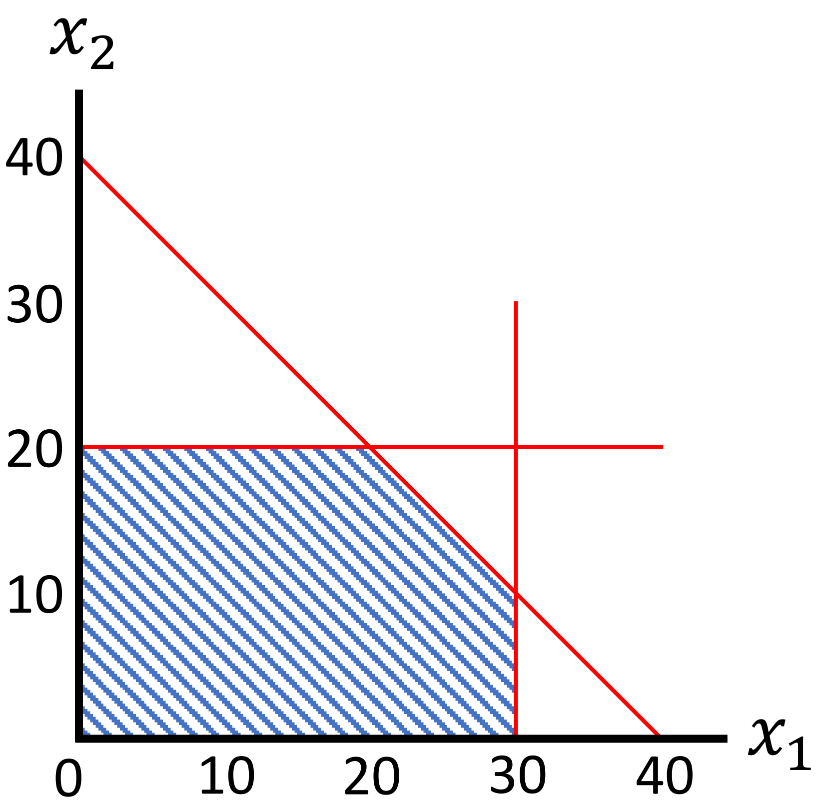


Feasible Region

(area where *all* valid solutions lie)

Example



$x_1, x_2 \in \mathbb{R}$

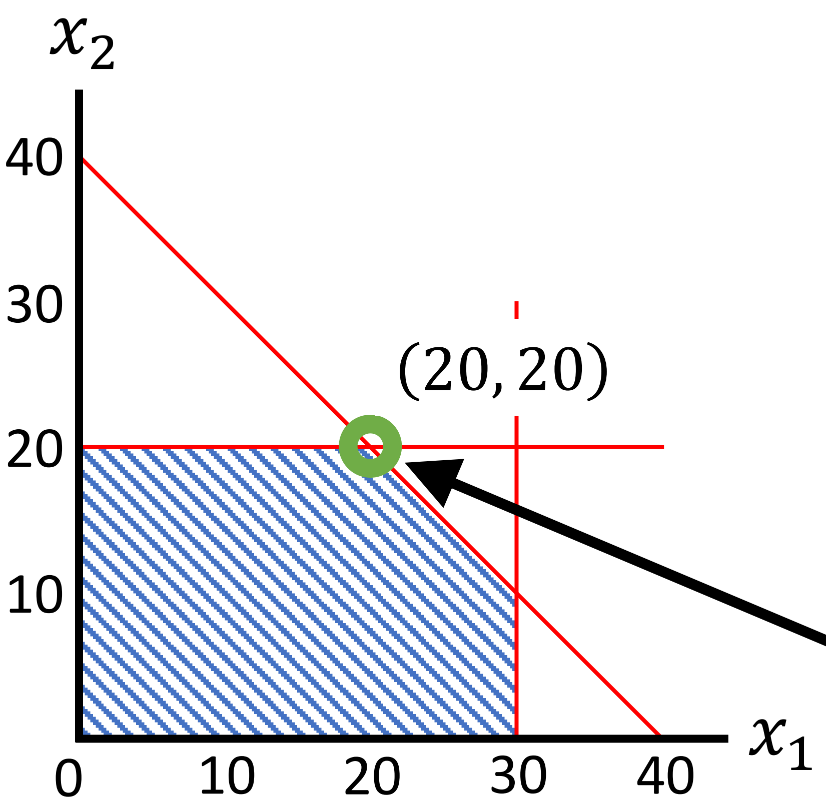
Objective: $\max 100x_1 + 300x_2$

Subject to:

$$x_1 \le 30$$
$$x_2 \le 20$$
$$x_1 + x_2 \le 40$$
$$x_1, x_2 \ge 0$$

What is the optimal value?

Example



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Subject to:
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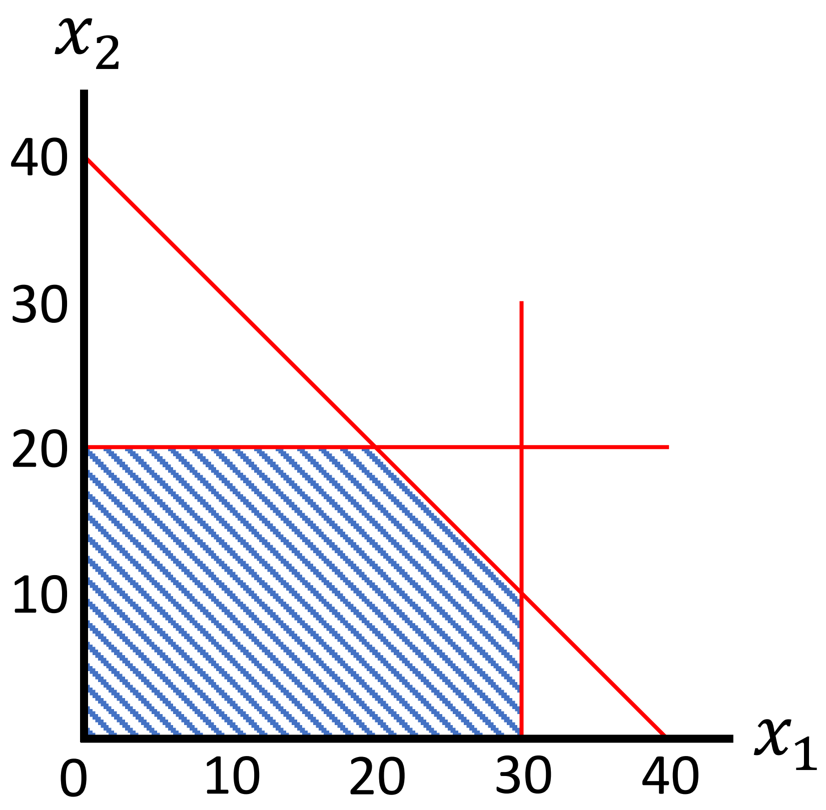
$\text{obj} = 100 * 20 + 300 * 20 = 8000$

Optimal Value



Objective: $\max f(x_1, x_2)$

Subject to: $c_1(x_1, x_2)$

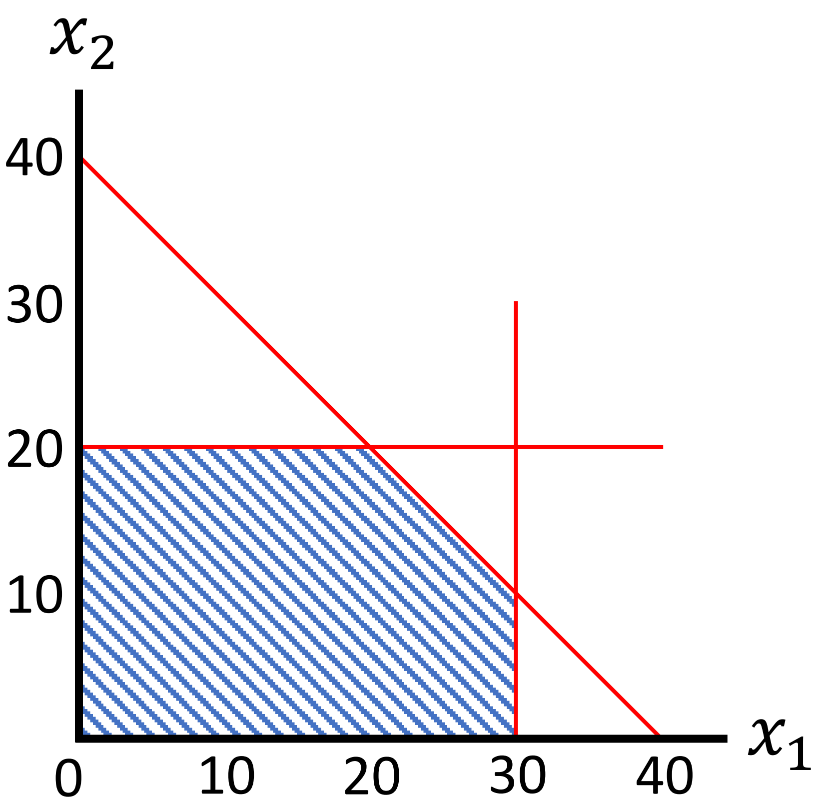
$c_2(x_1, x_2)$

$\vdots$

$c_n(x_1, x_2)$

How can we efficiently find optimal solutions?

Optimal Value



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Subject to: $c_1(x_1, x_2)$

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$c_n(x_1, x_2)$

How can we efficiently find optimal solutions?

Identify two key properties of optimal solutions:

1. ?
2. ?

Optimal Value

Objective: $\max f(x_1, x_2)$

Subject to: $c_1(x_1, x_2)$

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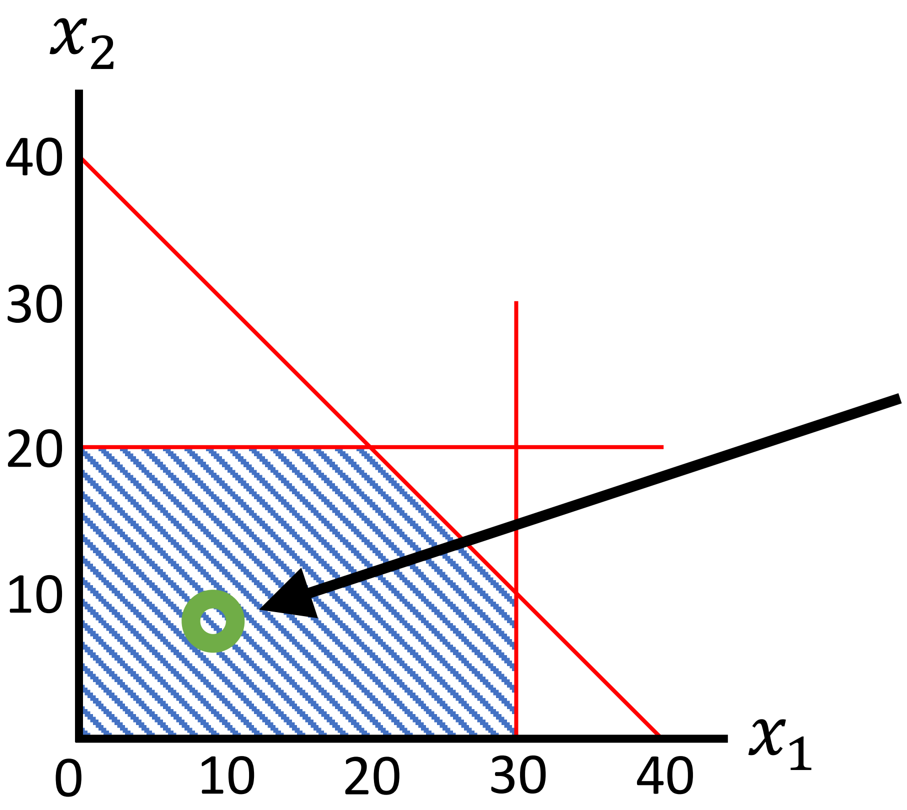
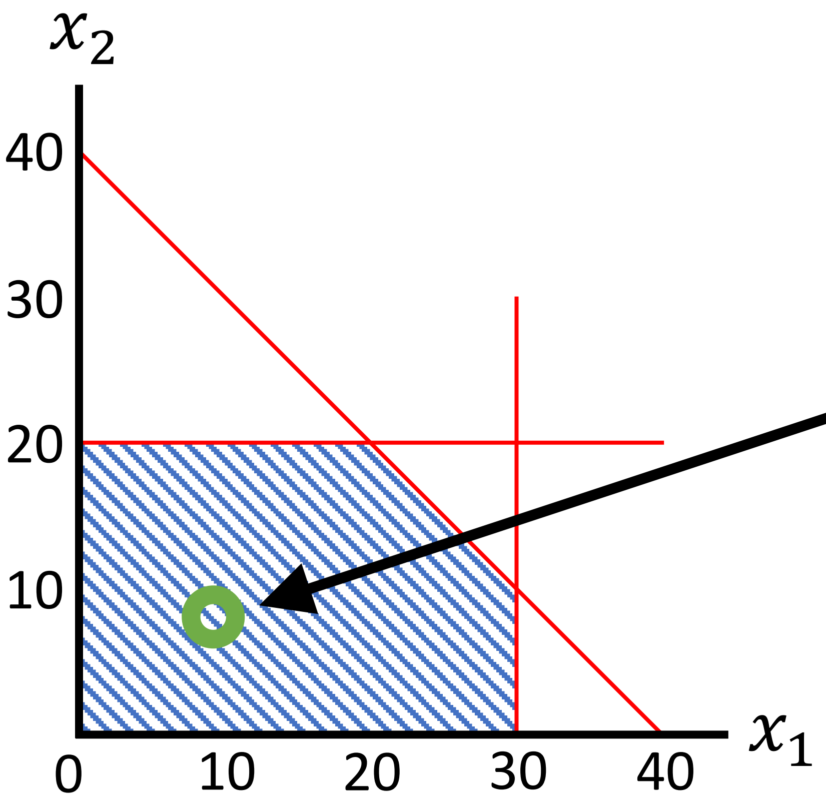


Could this ever be a maximum value of the objective function?

Optimal Value



Objective: $\max f(x_1, x_2)$

Subject to: $c_1(x_1, x_2)$

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$\vdots$

$c_n(x_1, x_2)$

Could this ever be a maximum value of the objective function?

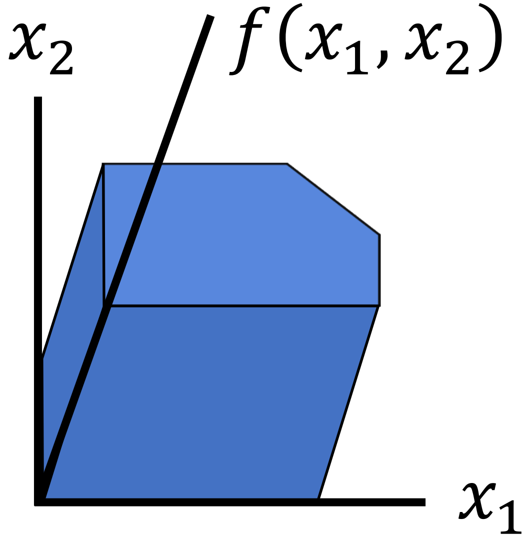
Yes, if $f(x_1, x_2) = \text{constant}$.

Objective: $\max 5$

Subject to: $c_1(x_1, x_2)$

$\vdots$

Optimal Value



Objective: $\max f(x_1, x_2)$

Subject to: $c_1(x_1, x_2)$

$c_2(x_1, x_2)$

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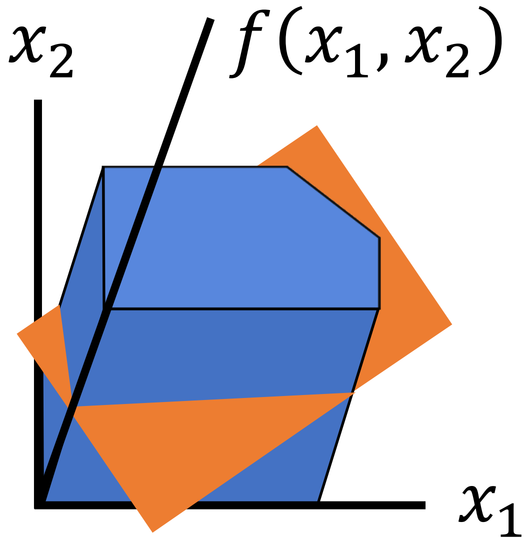
Optimal Value



Objective: $\max f(x_1, x_2)$

Subject to: $c_1(x_1, x_2)$

$c_2(x_1, x_2)$

$\vdots$

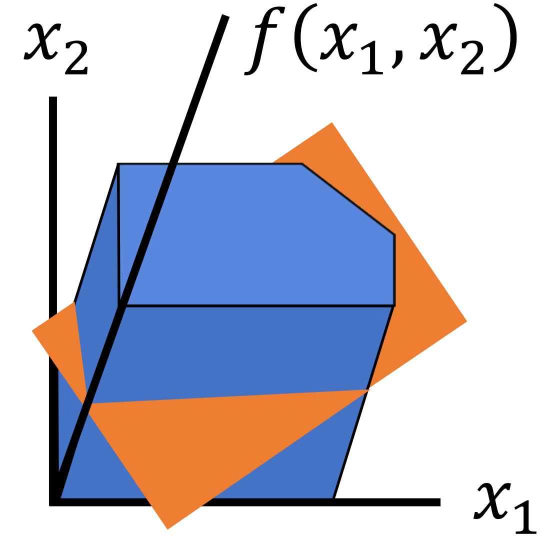
$c_n(x_1, x_2)$

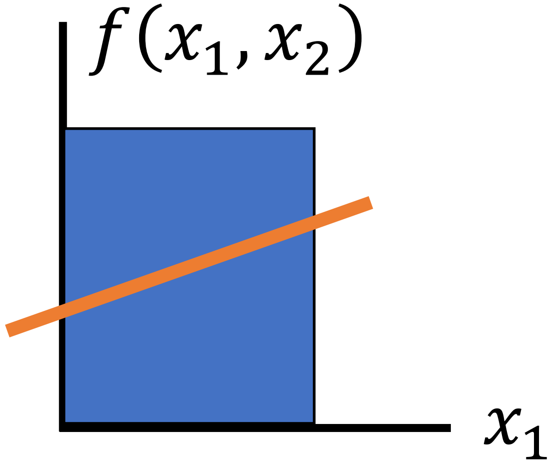
Could this ever be a maximum value of the objective function?

Yes, if $f(x_1, x_2) = \text{constant}$.

$f(x_1, x_2)$ is a plane

Optimal Value





Objective: $\max f(x_1, x_2)$

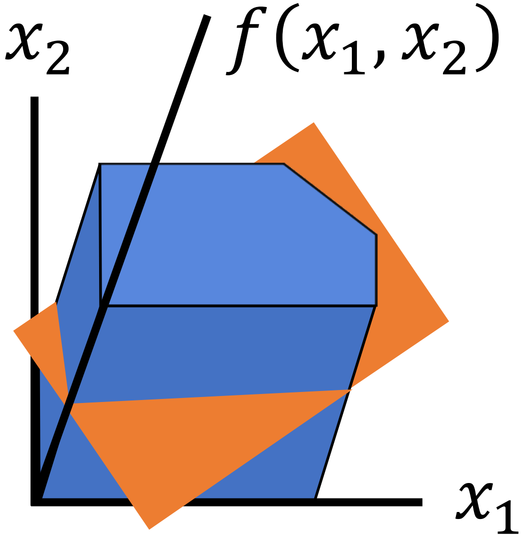
Subject to:
$$\begin{array}{c} c_1(x_1, x_2) \\ c_2(x_1, x_2) \\ \vdots \\ c_n(x_1, x_2) \end{array}$$

Could this ever be a maximum value of the objective function?

Yes, if $f(x_1, x_2) = \text{constant}$.

$f(x_1, x_2)$ is a plane $\Longrightarrow$ a max/min of $f(x_1, x_2)$ occurs on the boundary of the feasible region.

Optimal Value



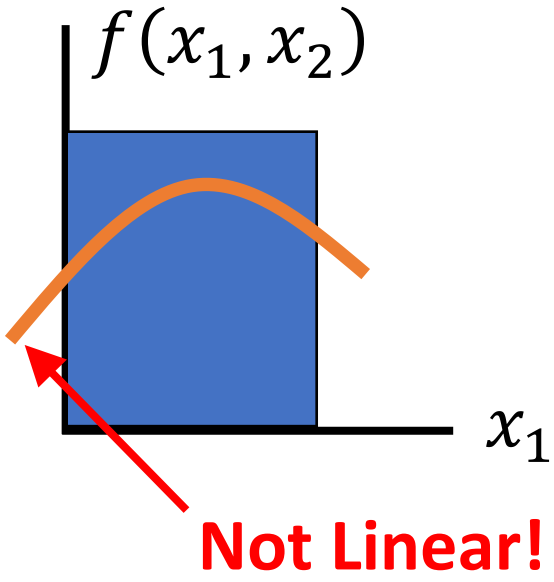
Objective: $\max f(x_1, x_2)$

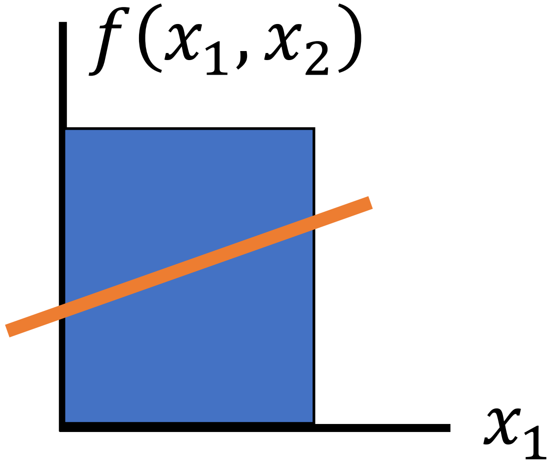
Subject to: $c_1(x_1, x_2)$

$c_2(x_1, x_2)$

$\vdots$

$c_n(x_1, x_2)$





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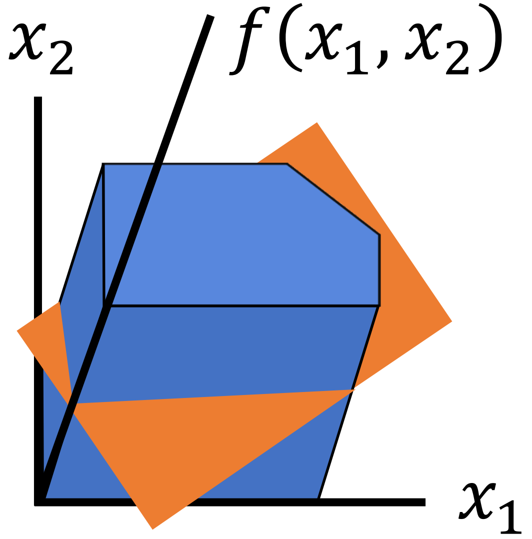

Objective: $\max f(x_1, x_2)$

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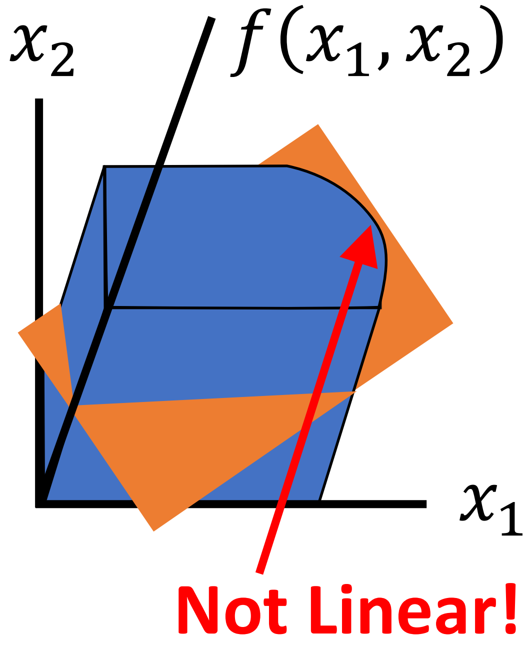

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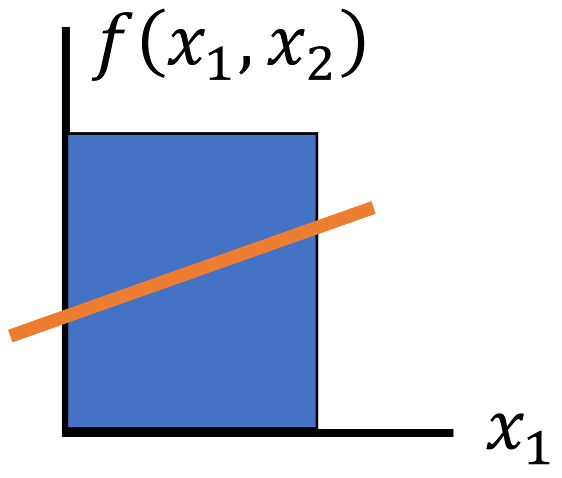
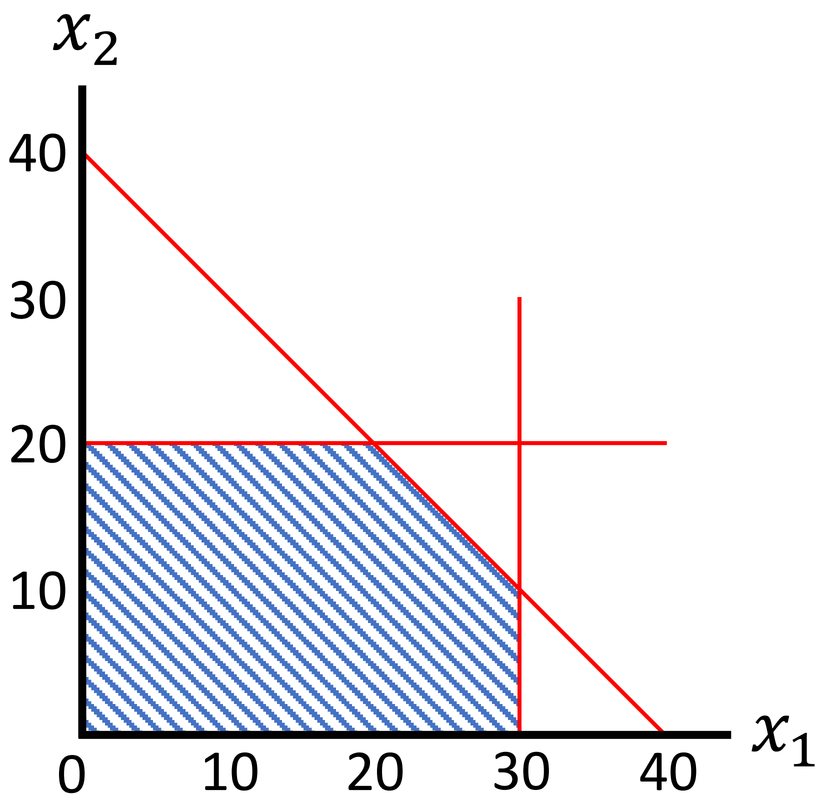


Yes, if $f(x_1, x_2) = \text{constant}$.

$f(x_1, x_2)$ is a plane $\Longrightarrow$ a max/min of $f(x_1, x_2)$ occurs on the boundary of the feasible region. Since feasible region has linear boundaries, max/min must occur at a vertex in the feasible region.

Optimal Value



Objective: $\max f(x_1, x_2)$

Subject to: $c_1(x_1, x_2)$

$c_2(x_1, x_2)$

$\vdots$

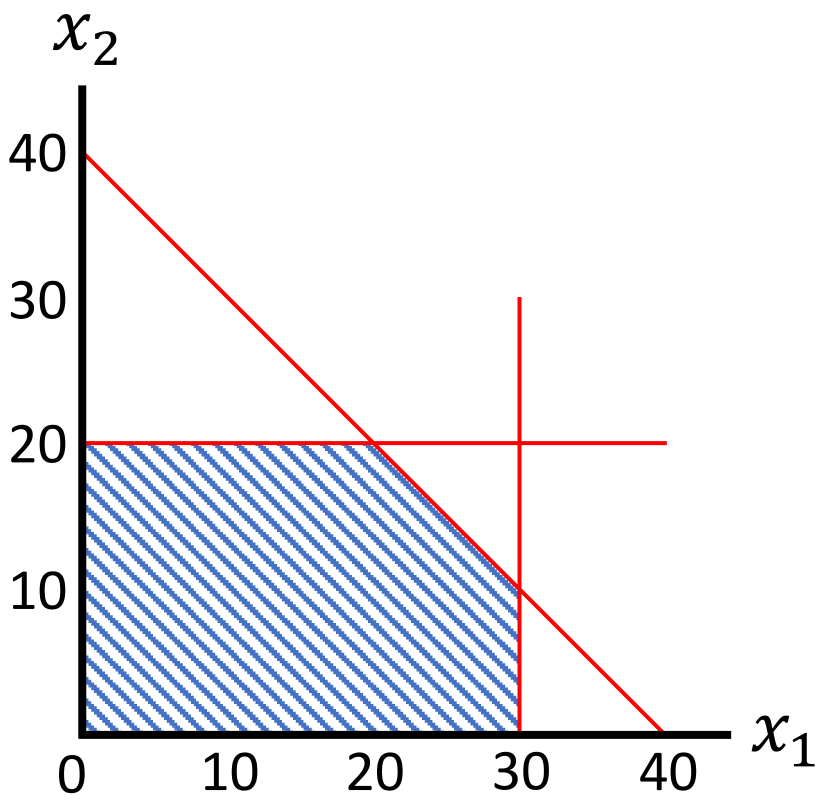
$c_n(x_1, x_2)$

How can we efficiently find optimal solutions?

Identify two key properties of optimal solutions:

1. Optimal value occurs at a vertex.
2. ?

Optimal Value



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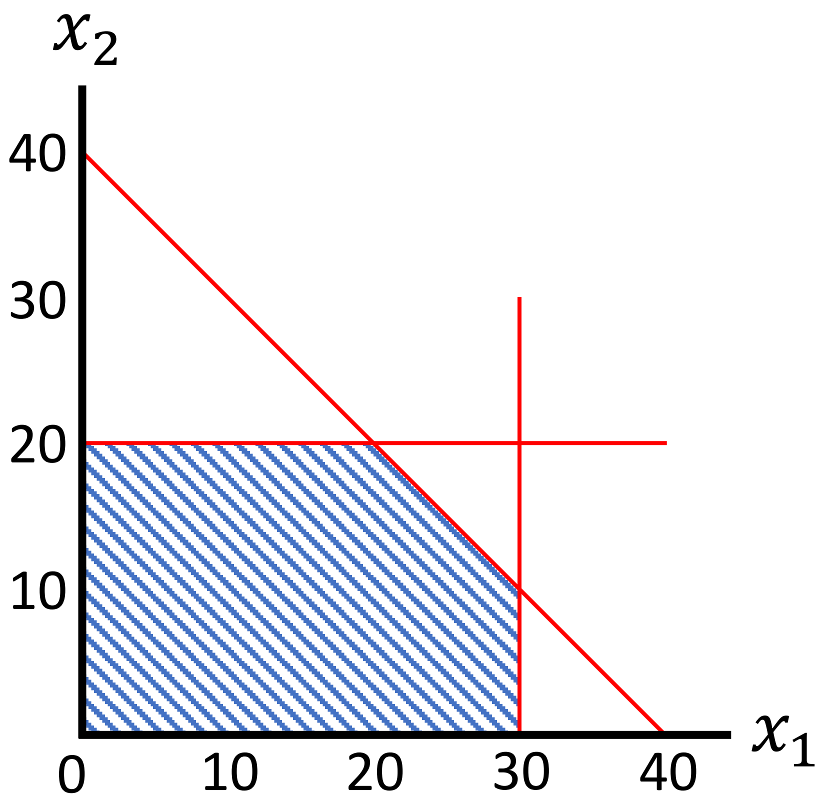
Is there a relationship between a local max/min and a global max/min?

Optimal Value



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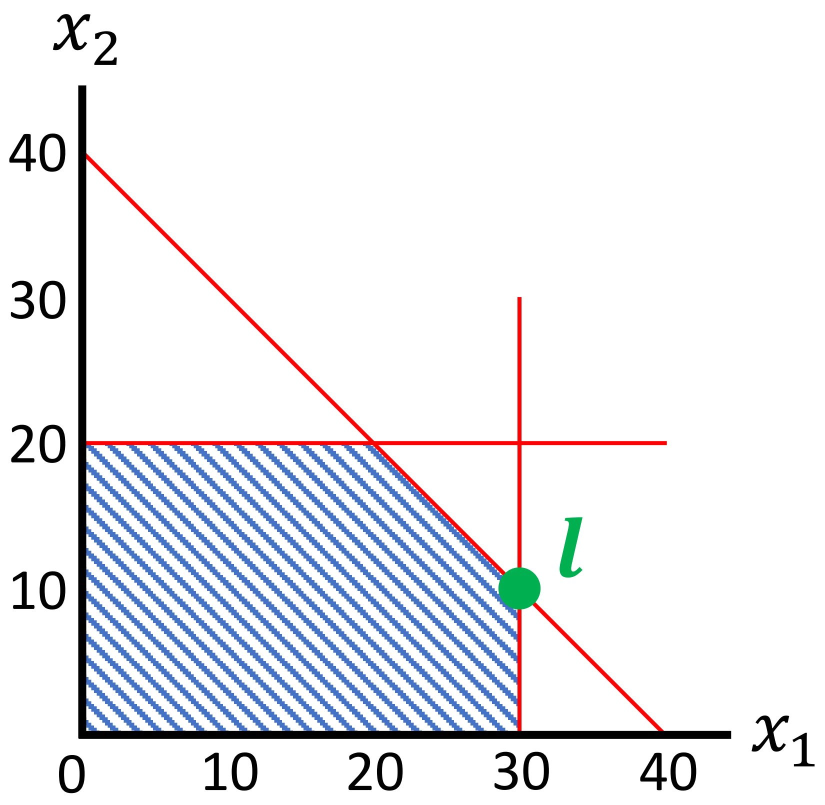
Is there a relationship between a local max/min and a global max/min? local max/min = global max/min.

Optimal Value



Objective: $\max f(x_1, x_2)$

Subject to:
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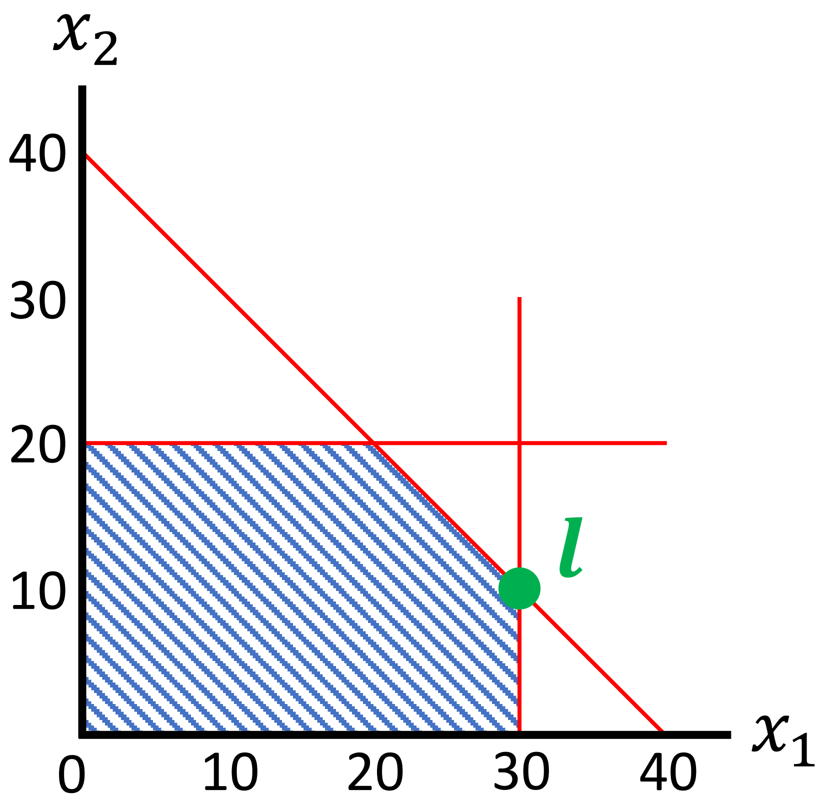
$$\vdots$$
$$c_n(x_1, x_2)$$

Is there a relationship between a local max/min and a global max/min? local max/min = global max/min.

local max ⇒ ?

Optimal Value



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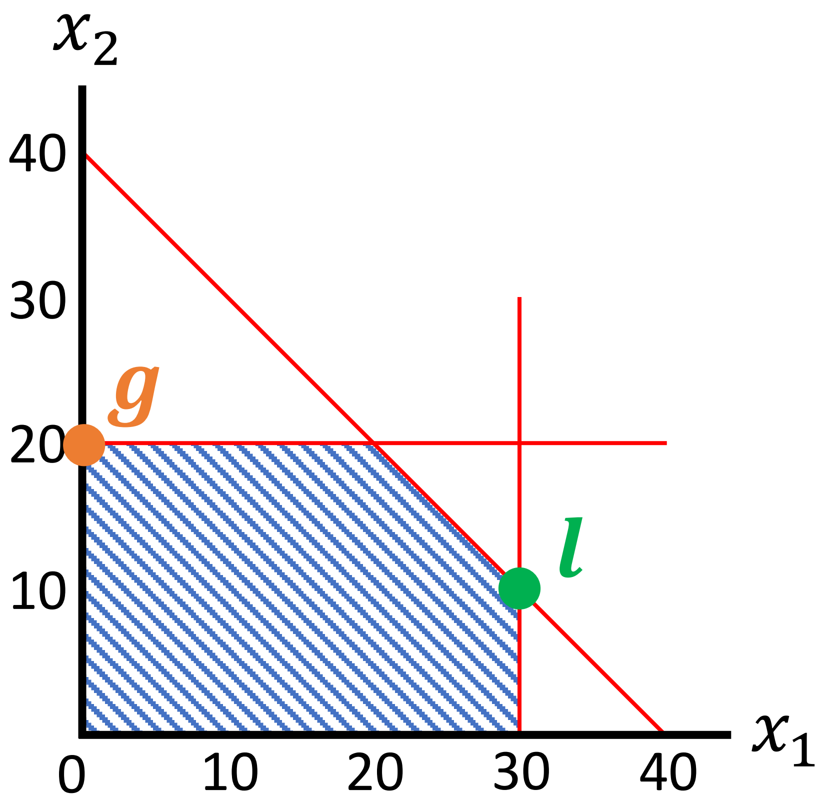
Is there a relationship between a local max/min and a global max/min? local max/min = global max/min.

local max ⇒ all points in ε-neighborhood of l have lower objective values.

Optimal Value



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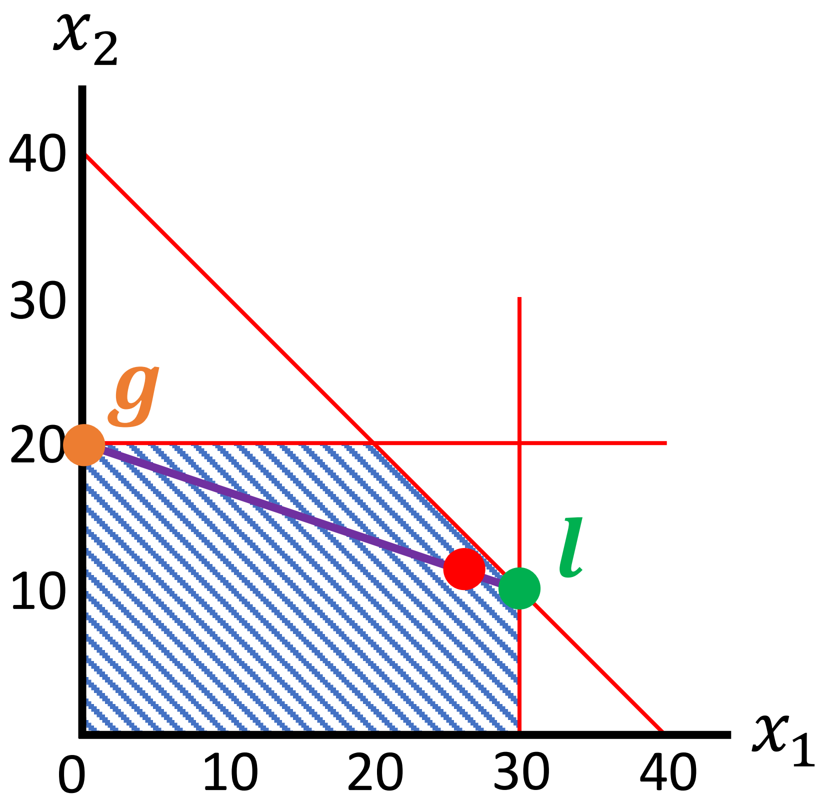
Is there a relationship between a local max/min and a global max/min? local max/min = global max/min.

local max $\Rightarrow$ all points in ε-neighborhood of l have lower objective values.

Let g be global max.

Optimal Value



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Is there a relationship between a local max/min and a global max/min? local max/min = global max/min.

local max $\Rightarrow$ all points in ε-neighborhood of l have lower objective values.

Let g be global max. Some point in ε-nbhd lies on the line between l and g and …

Optimal Value

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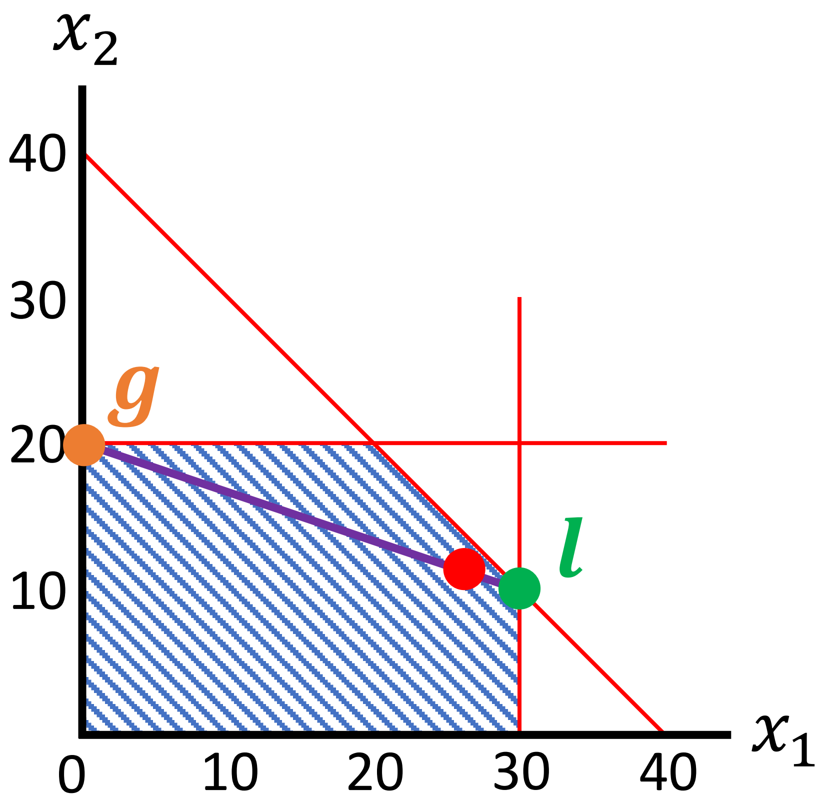
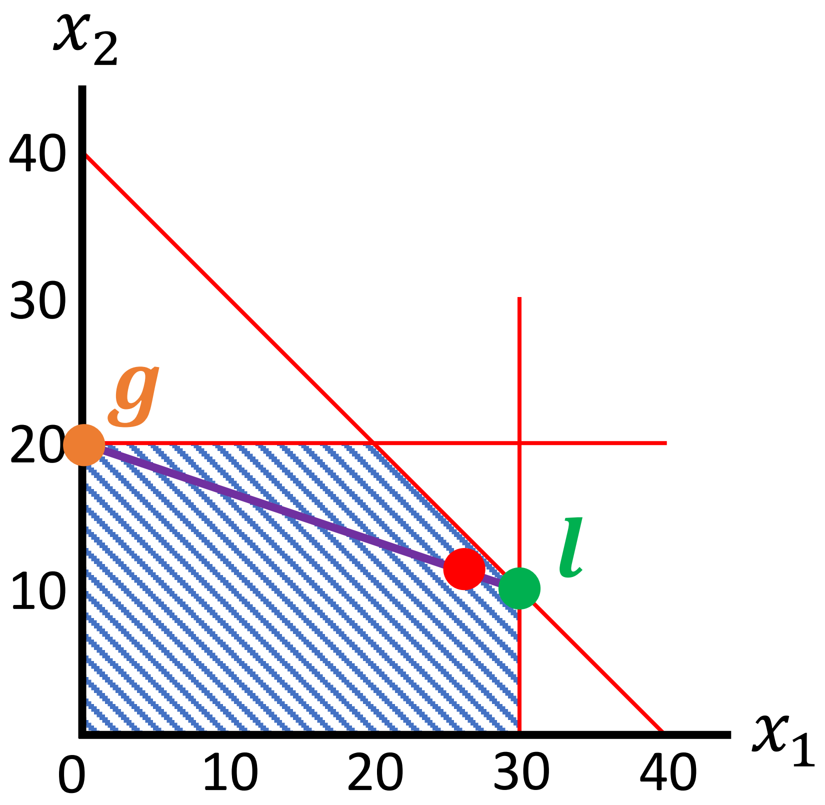
Subject to:	$c_1(x_1, x_2)$
	$c_2(x_1, x_2)$
	$\vdots$
	$c_n(x_1, x_2)$



Is there a relationship between a local max/min and a global max/min? local max/min = global max/min.

local max ⇒ all points in ε-neighborhood of l have lower objective values.
Let g be global max. Some point in ε-nbhd lies on the line between l and g and all points on that line are feasible (?).

Optimal Value



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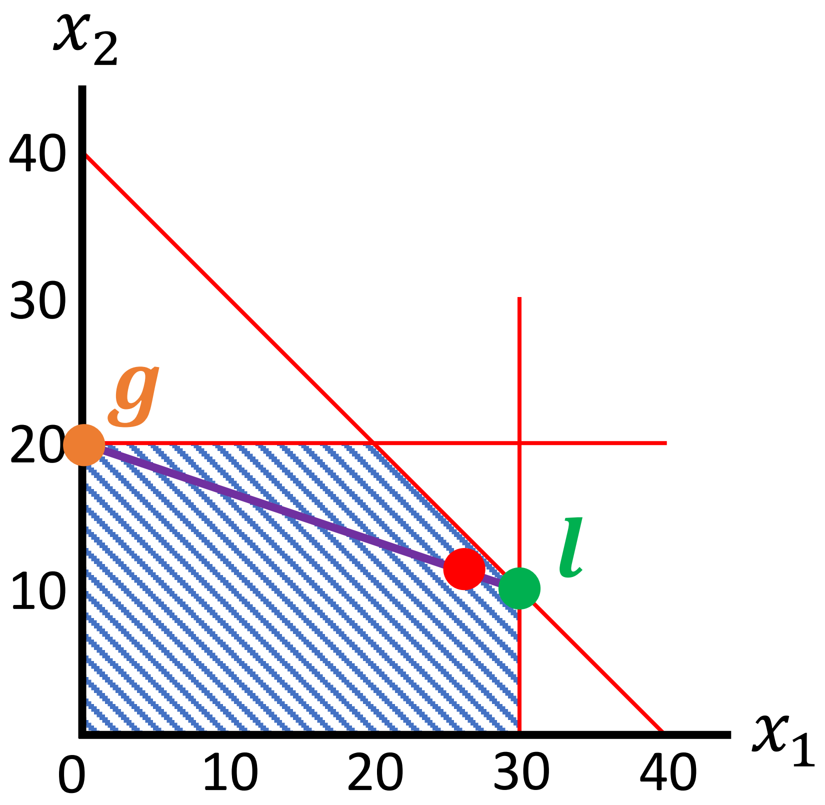
Let g be global max. Some point in ε-nbhd lies on the line between l and g and all points on that line are feasible (convex feasible region).

Optimal Value



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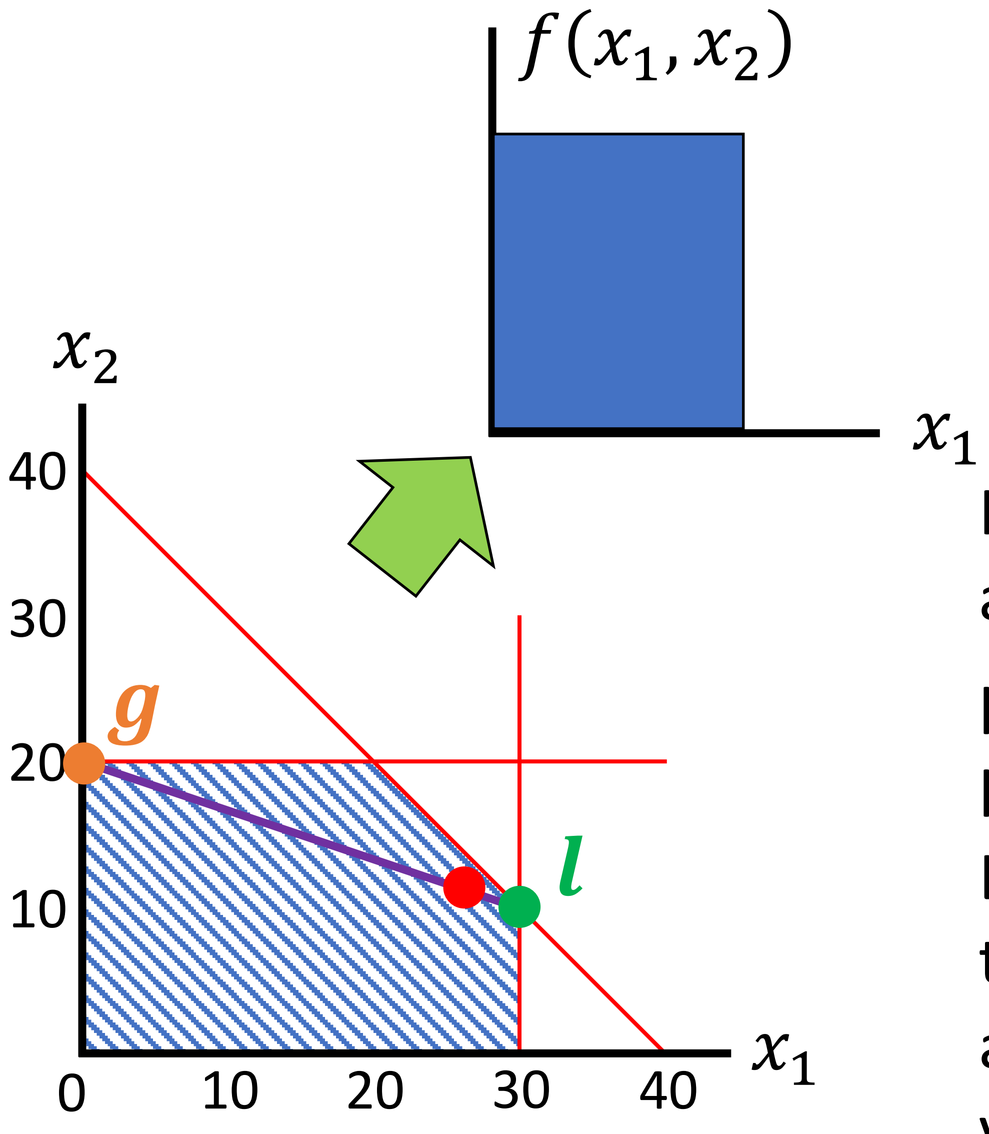
Let g be global max. Some point in ε-nbhd lies on the line between l and g and all points on that line are feasible (convex feasible region). Thus, objective values of the line is a line on objective hyperplane...

Optimal Value



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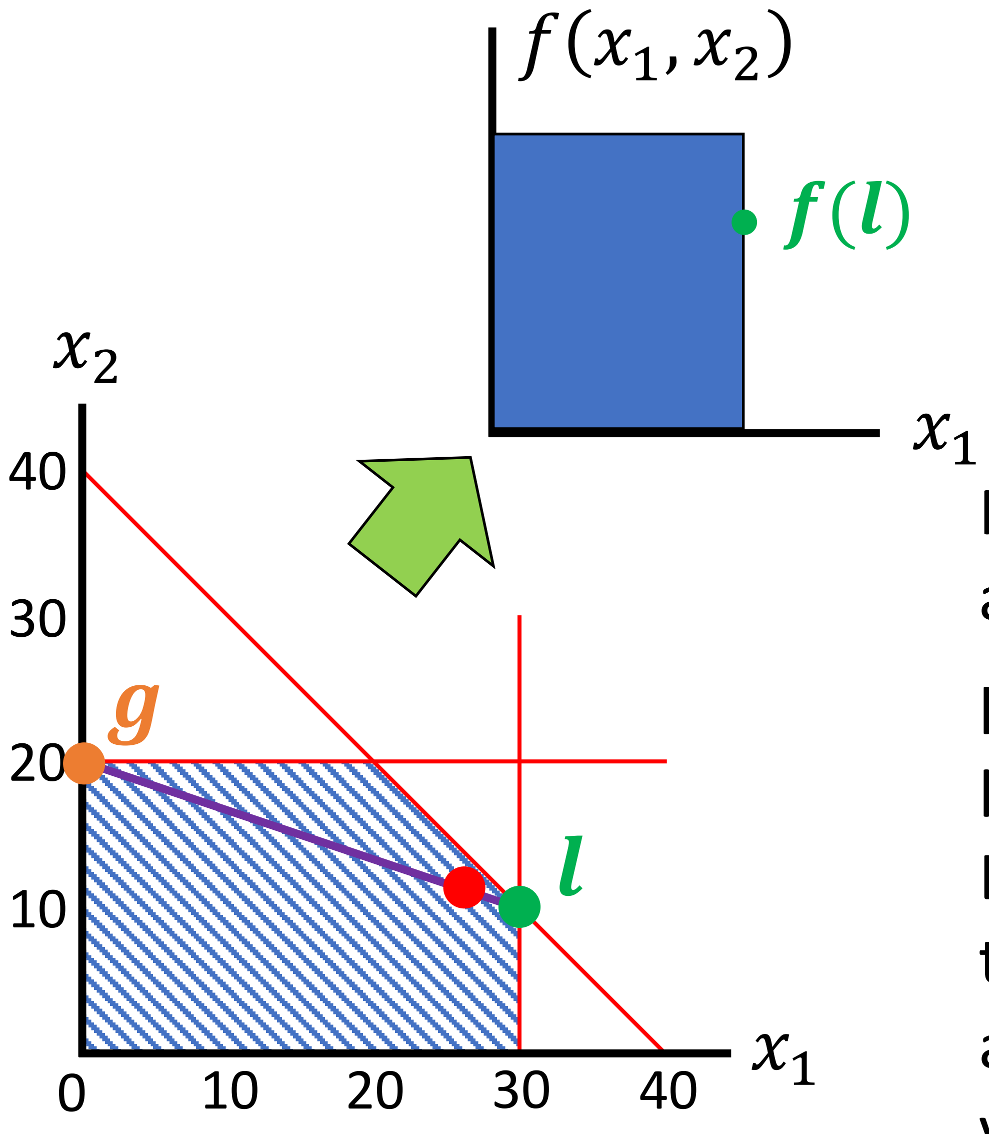

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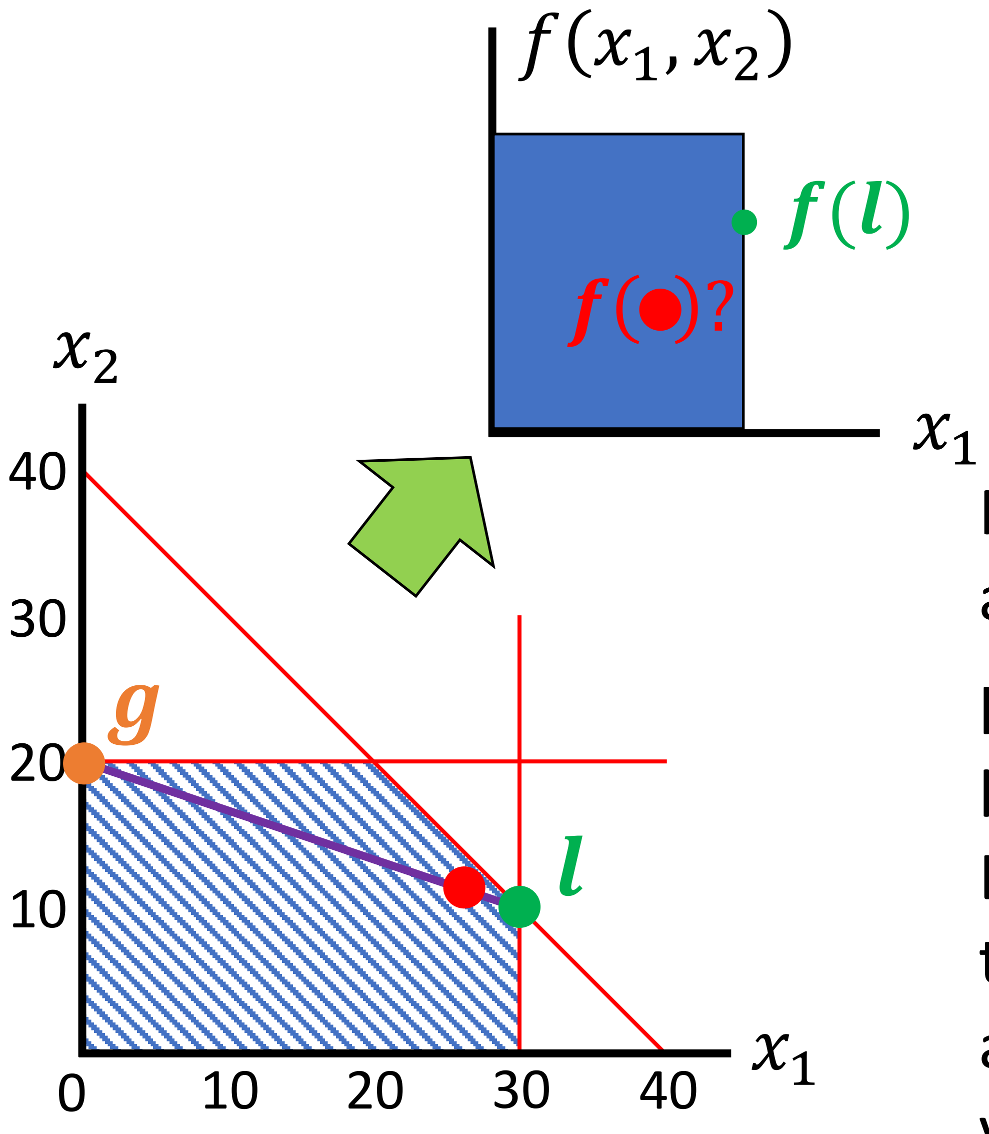

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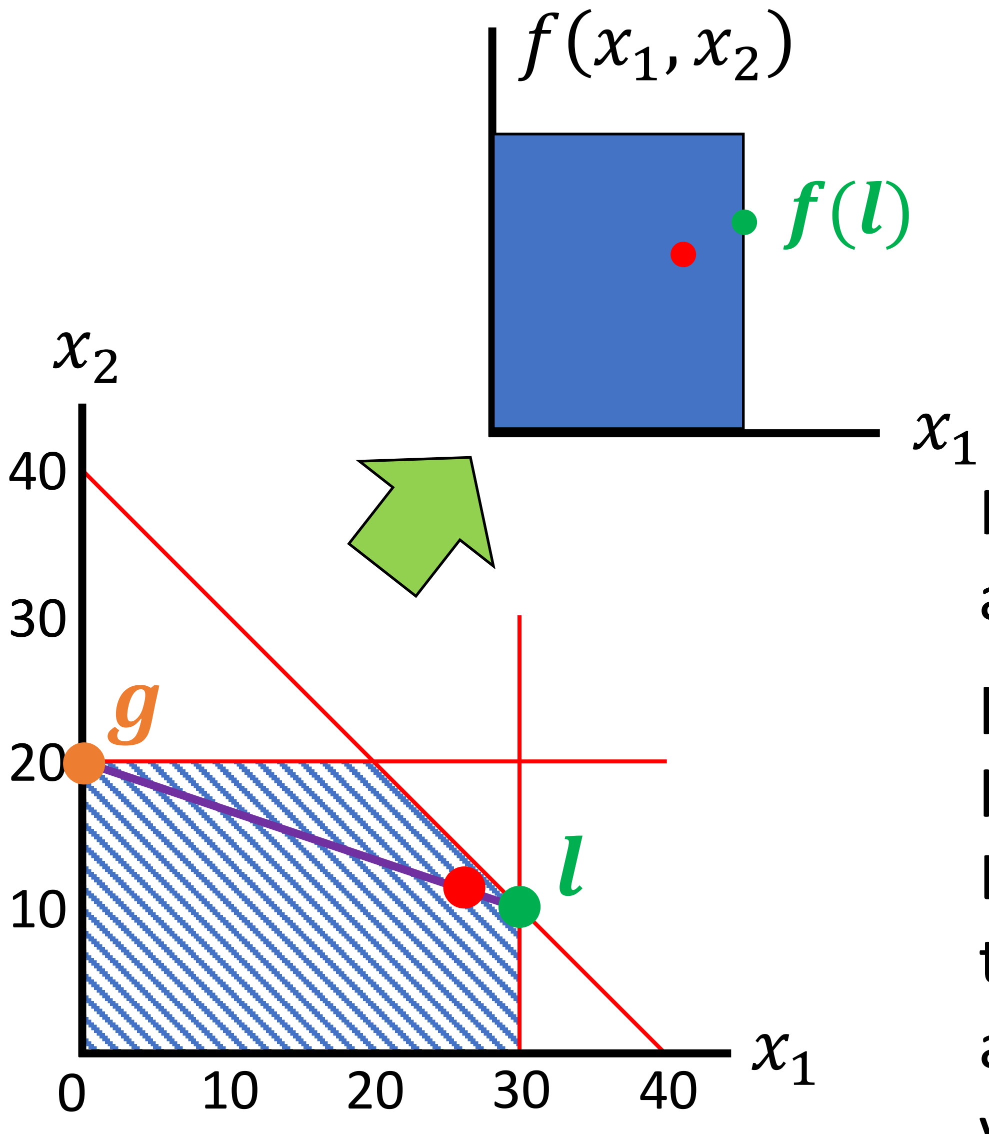

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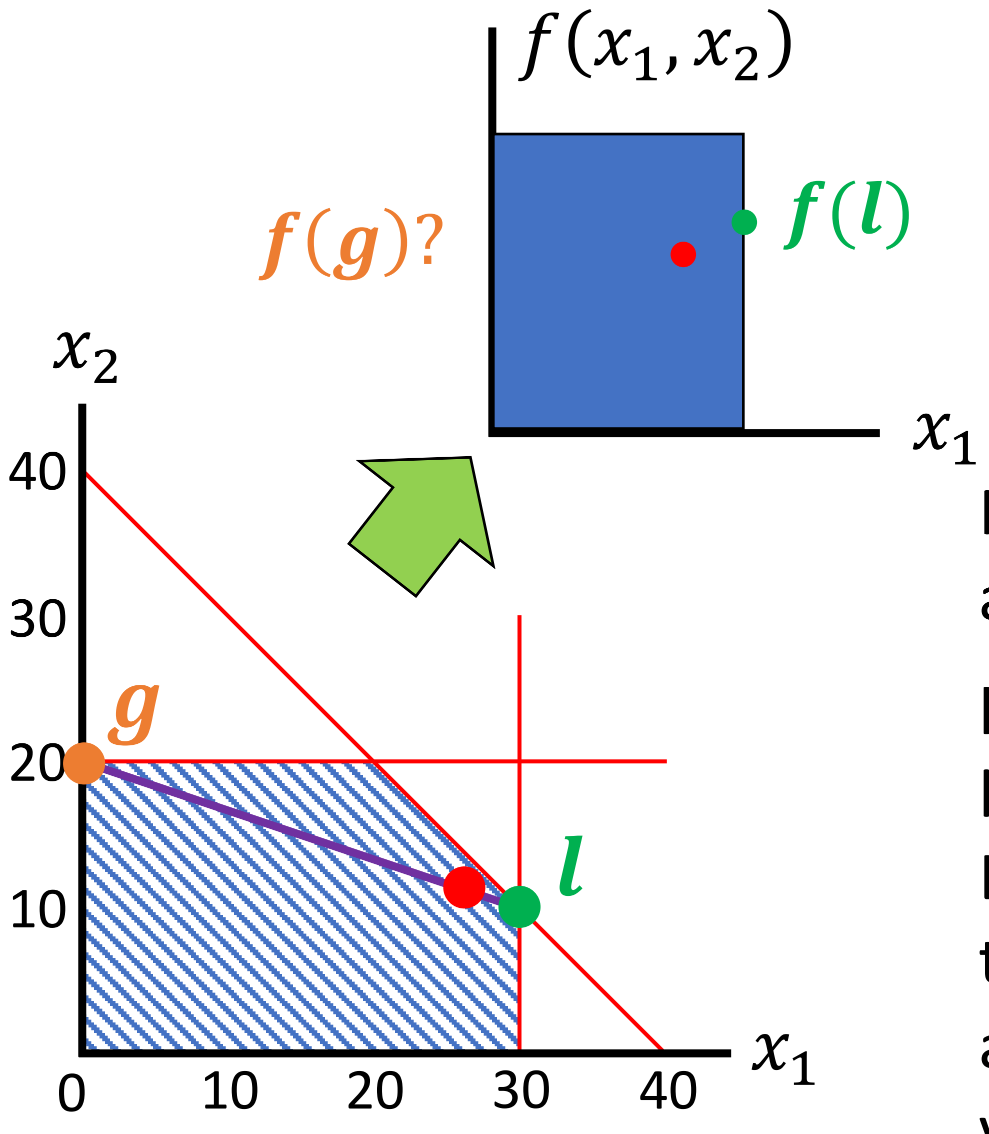

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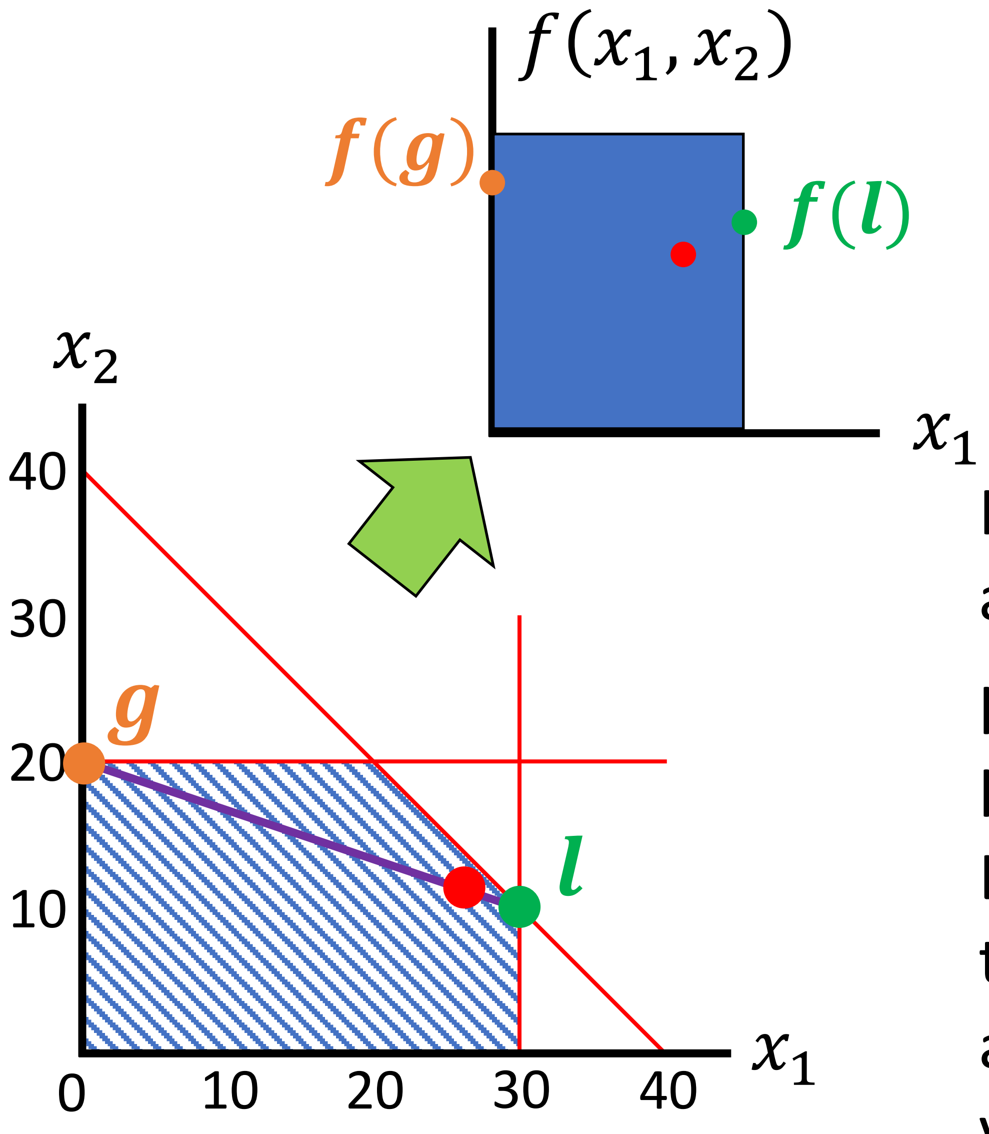
Optimal Value



Objective: $\max f(x_1, x_2)$

Subject to: $c_1(x_1, x_2)$

$c_2(x_1, x_2)$

$\vdots$

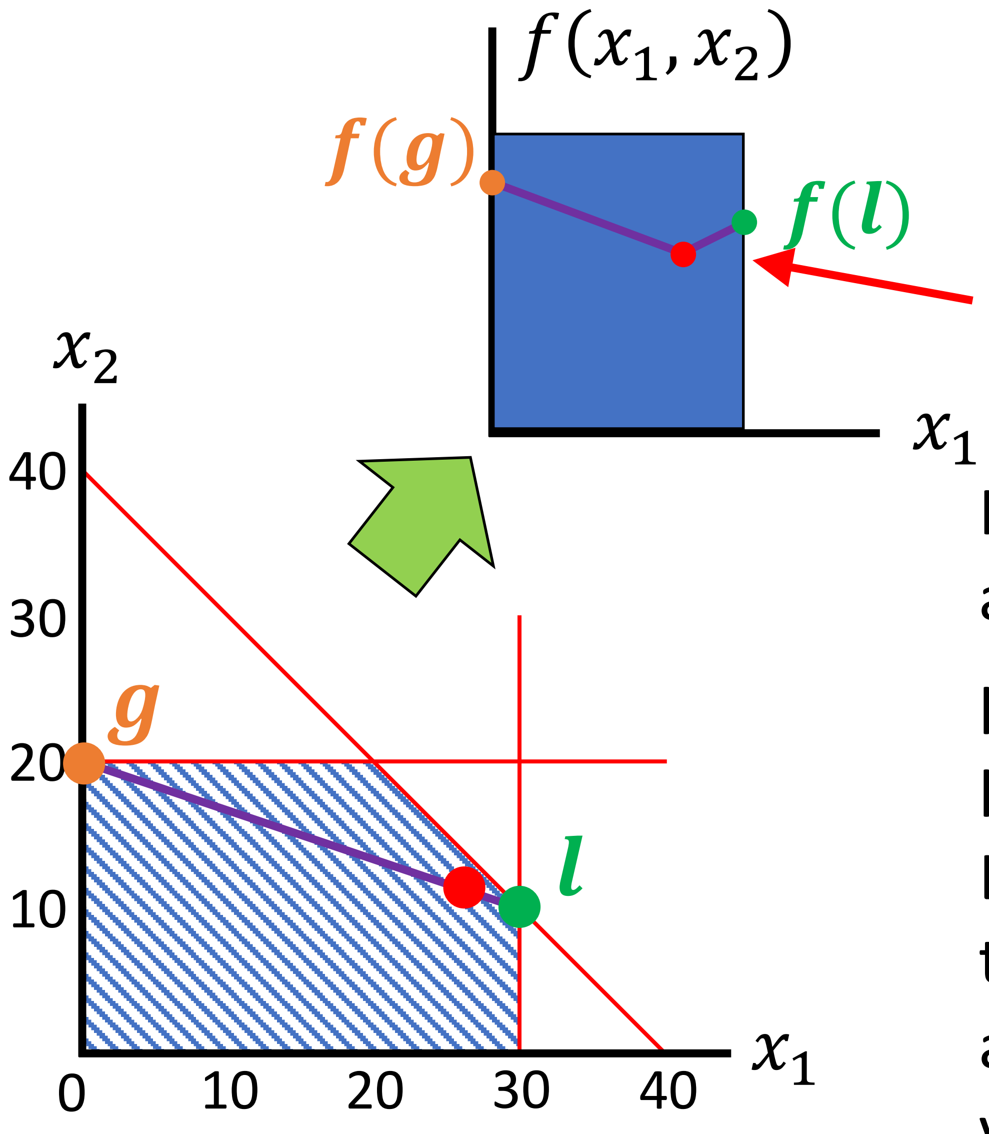
$c_n(x_1, x_2)$

Is there a relationship between a local max/min and a global max/min? local max/min = global max/min.

local max ⇒ all points in ε-neighborhood of l have lower objective values.

Let g be global max. Some point in ε-nbhd lies on the line between l and g and all points on that line are feasible (convex feasible region). Thus, objective values of the line is a line on objective hyperplane...

Optimal Value



Not Linear!

Objective: $\max f(x_1, x_2)$

Subject to: $c_1(x_1, x_2)$

$c_2(x_1, x_2)$

$\vdots$

$c_n(x_1, x_2)$

Is there a relationship between a local max/min and a global max/min? local max/min = global max/min.

local max $\Rightarrow$ all points in ε-neighborhood of l have lower objective values.

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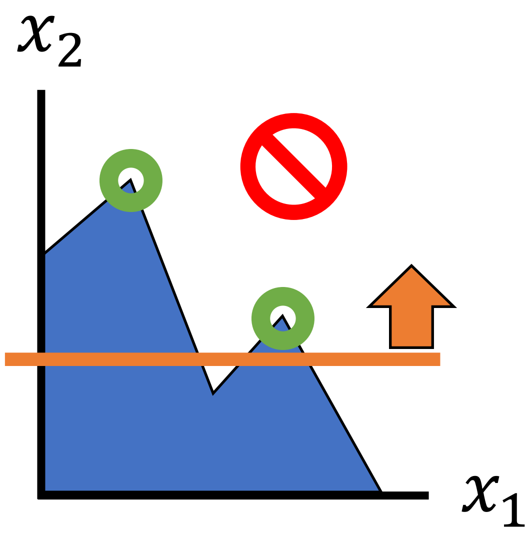
Optimal Value



The only way for local optimum $\neq$ global optimum ***and*** objective be linear is for feasible region to not be convex.

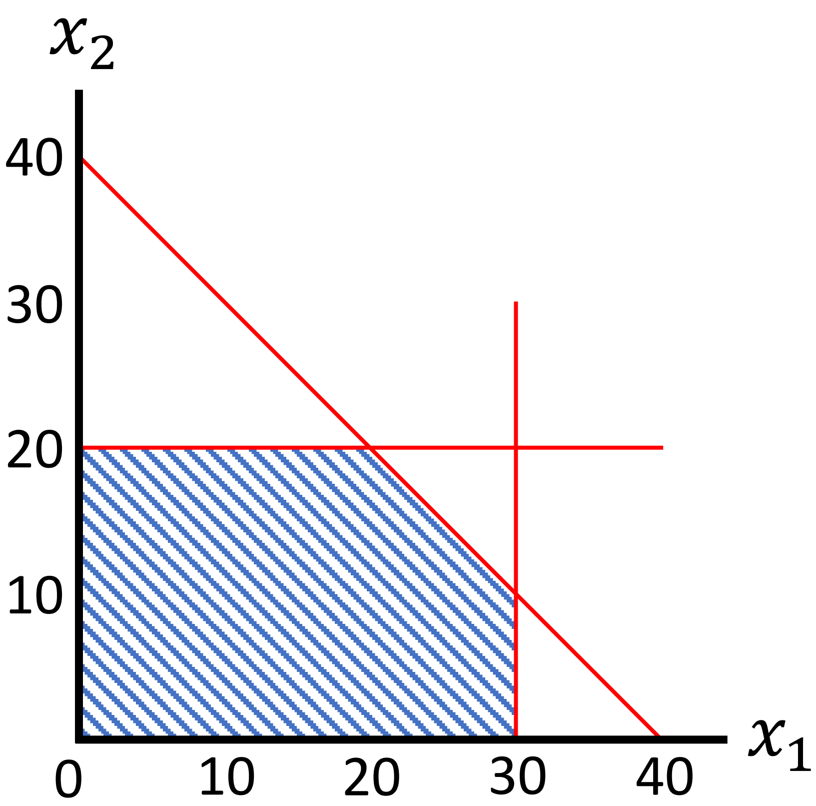
Objective: $\max f(x_1, x_2)$

Subject to:
$$
\begin{aligned}
&c_1(x_1, x_2)\\
&c_2(x_1, x_2)\\
&\vdots\\
&c_n(x_1, x_2)
\end{aligned}
$$

Is there a relationship between a local max/min and a global max/min? local max/min = global max/min.

local max $\Rightarrow$ all points in ε-neighborhood of l have lower objective values.
Let g be global max. Some point in ε-nbhd lies on the line between l and g and all points on that line are feasible (convex feasible region). Thus, objective values of the line is a line on objective hyperplane...

Optimal Value



Objective: $\max f(x_1, x_2)$

Subject to: $c_1(x_1, x_2)$

$c_2(x_1, x_2)$

$\vdots$

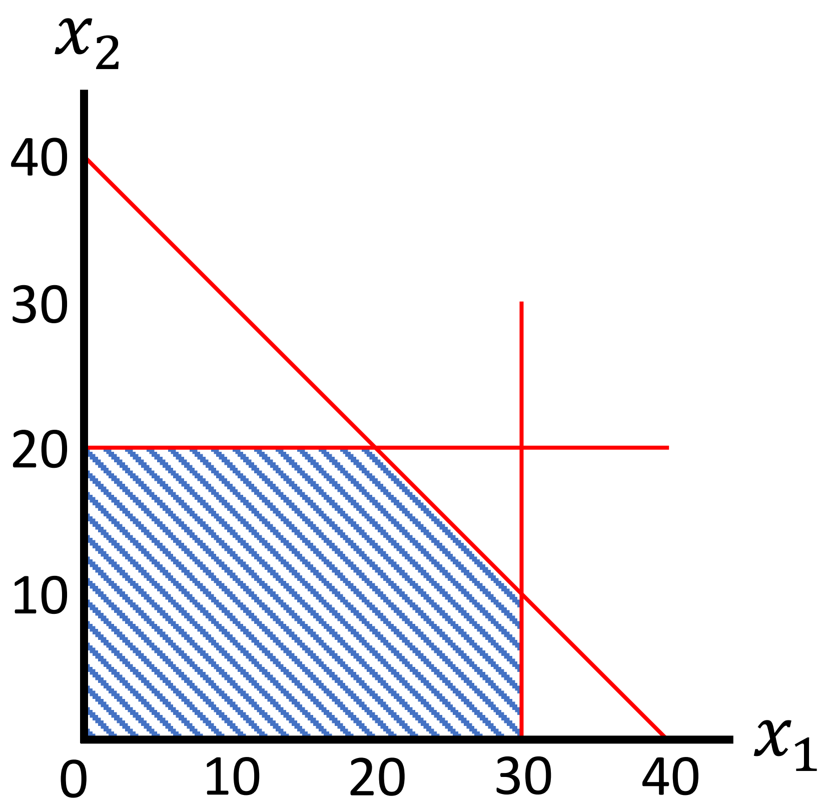
$c_n(x_1, x_2)$

How can we efficiently find optimal solutions?

Identify two key properties of optimal solutions:

1. Optimal value occurs at a vertex.
2. Local optimum is global optimum.

Optimal Value



Objective: $\max f(x_1, x_2)$

Subject to: $c_1(x_1, x_2)$

$c_2(x_1, x_2)$

$\vdots$

$c_n(x_1, x_2)$

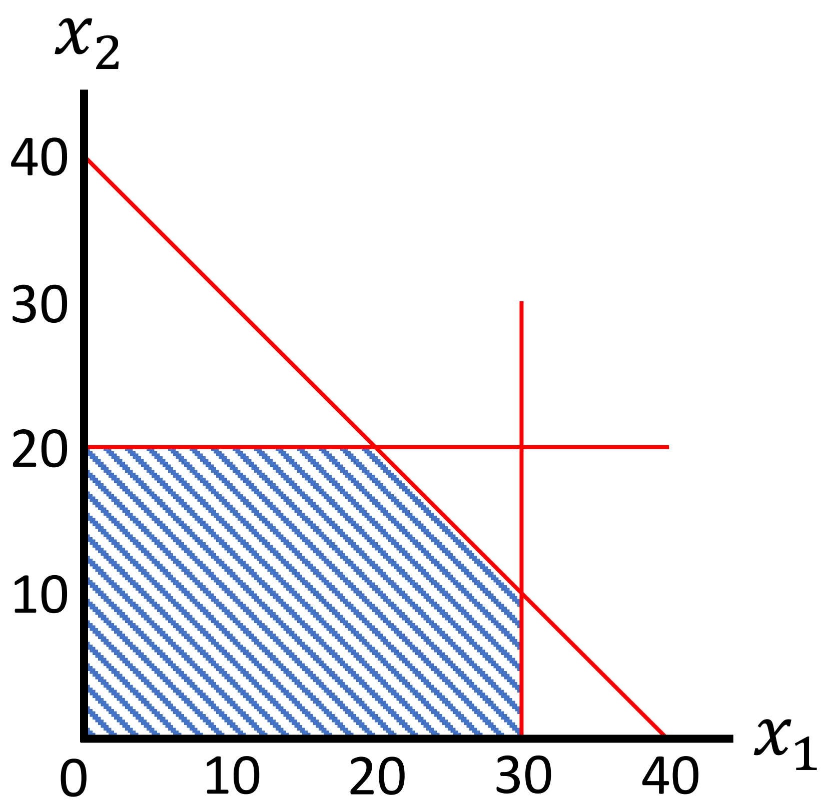
Properties of optimal solutions:

1. Optimal value occurs at a vertex.
2. Local optimum is global optimum.

Algorithm to find optimal solution:

Test each vertex in order until no neighbors have larger (or smaller) value.

Example



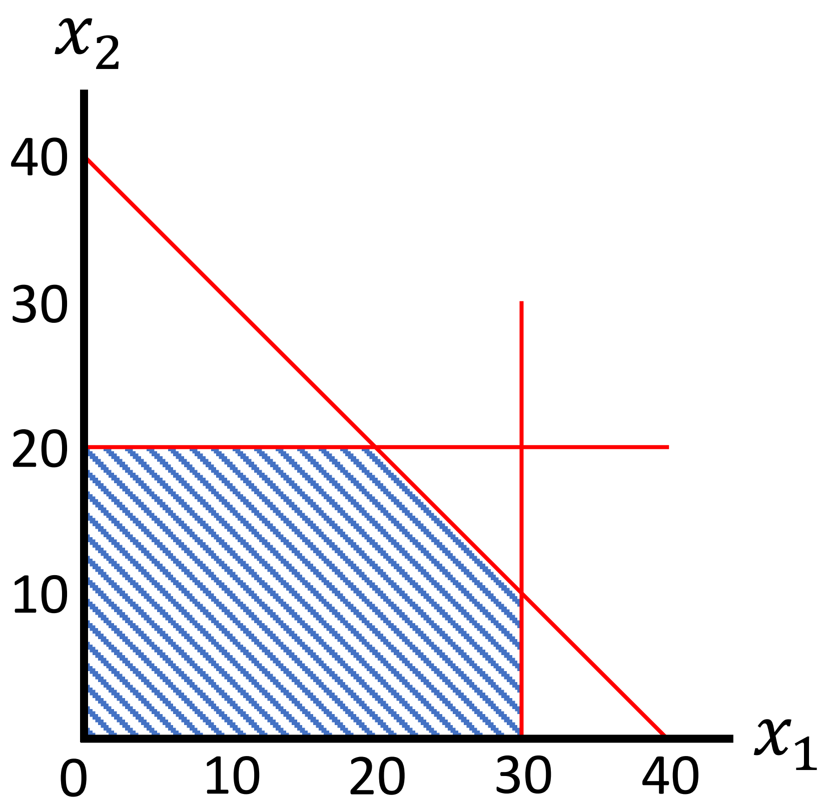
$x_1, x_2 \in \mathbb{R}$

Objective: $\max 100x_1 + 300x_2$

Subject to:

$$x_1 \leq 30$$
$$x_2 \leq 20$$
$$x_1 + x_2 \leq 40$$
$$x_1, x_2 \geq 0$$

Example



$x_1, x_2, x_3 \in \mathbb{R}$

Objective: $\max 100x_1 + 300x_2 + 150x_3$

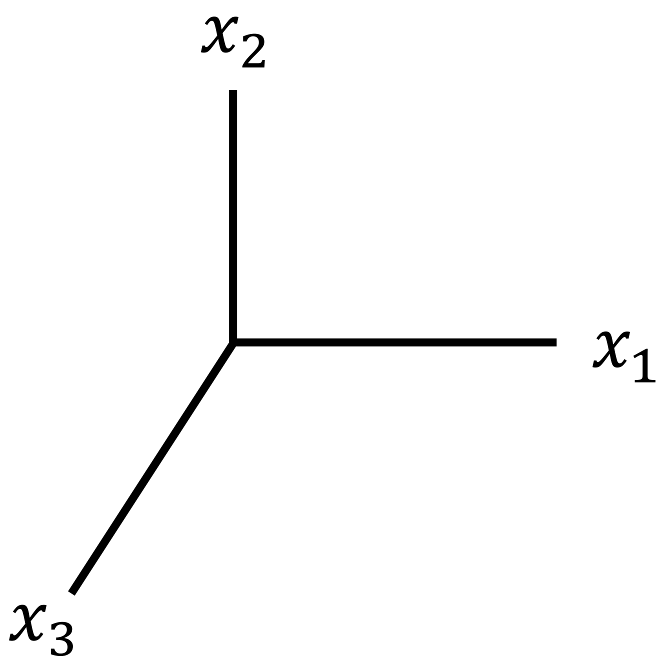
Subject to: $x_2 \leq 20$

$x_1 + x_2 + x_3 \leq 40$

$2x_1 + x_3 \leq 60$

$x_1, x_2, x_3 \geq 0$

Example



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Subject to:

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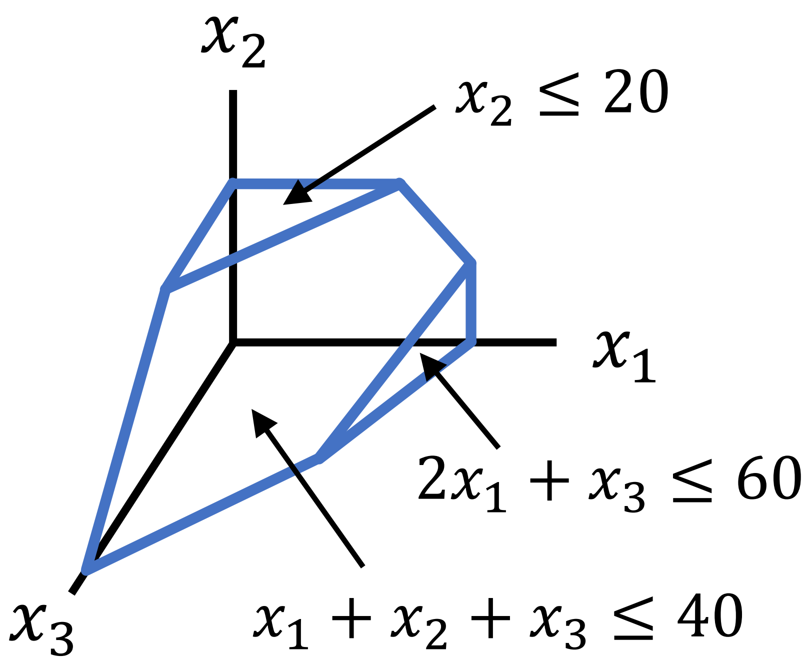
Example



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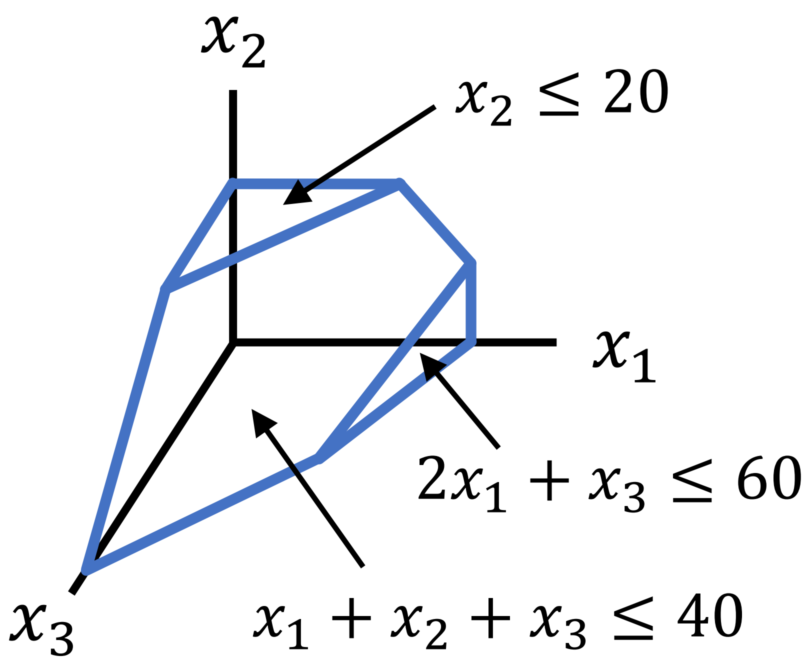
Example



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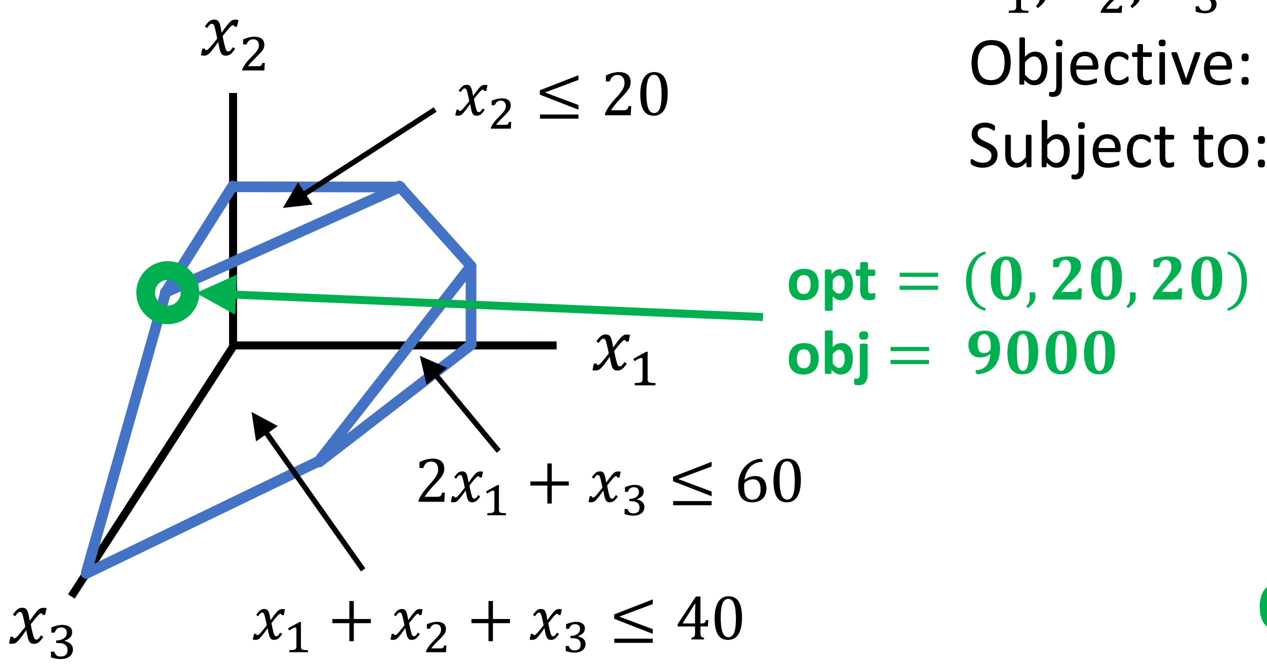
Objective: $\max 100x_1 + 300x_2 + 150x_3$

Subject to:

$$x_2 \le 20$$
$$x_1 + x_2 + x_3 \le 40$$
$$2x_1 + x_3 \le 60$$
$$x_1, x_2, x_3 \ge 0$$

Optimal Solution?

Example



$x_1, x_2, x_3 \in \mathbb{R}$

Objective: $\max 100x_1 + 300x_2 + 150x_3$

Subject to:

$$x_2 \leq 20$$
$$x_1 + x_2 + x_3 \leq 40$$
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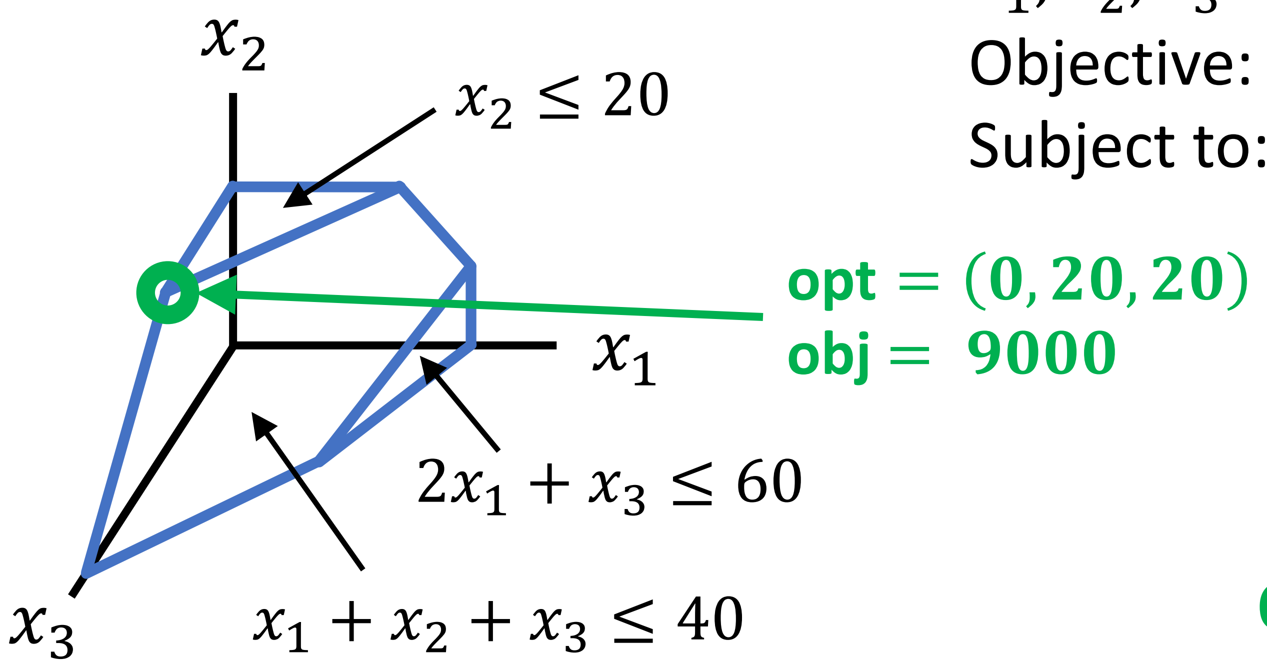
Optimal Solution?

Example



$x_2 \leq 20$

$x_1, x_2, x_3 \in \mathbb{R}$

Objective: $\max 100x_1 + 300x_2 + 150x_3$

Subject to:

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$$x_1 + x_2 + x_3 \leq 40$$
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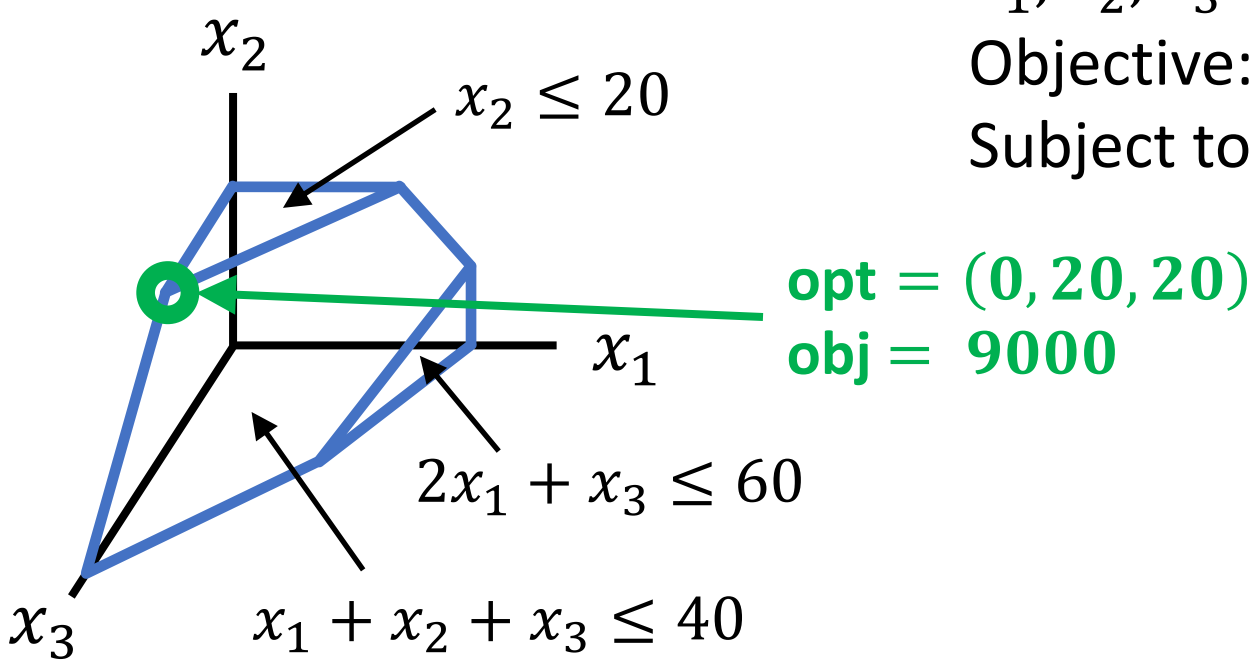
$$x_1, x_2, x_3 \geq 0$$

Optimal Solution?

Example



$x_1, x_2, x_3 \in \mathbb{R}$

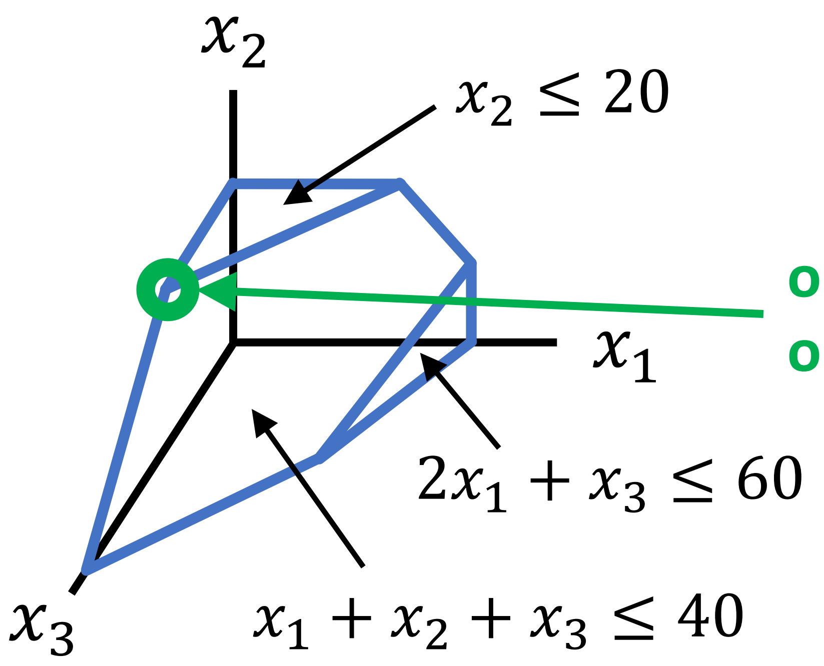
Objective: $\max 100x_1 + 300x_2 + 150x_3$

Subject to:
$$x_2 \leq 20$$
$$x_1 + x_2 + x_3 \leq 40$$
$$2x_1 + x_3 \leq 60$$
$$x_1, x_2, x_3 \geq 0$$

Optimal Solution?

$$x_2 \leq 20 \Longrightarrow 150x_2 \leq 3000$$

Example



opt $= (0, 20, 20)$
obj $= 9000$

$x_1, x_2, x_3 \in \mathbb{R}$

Objective: $\max 100x_1 + 300x_2 + 150x_3$

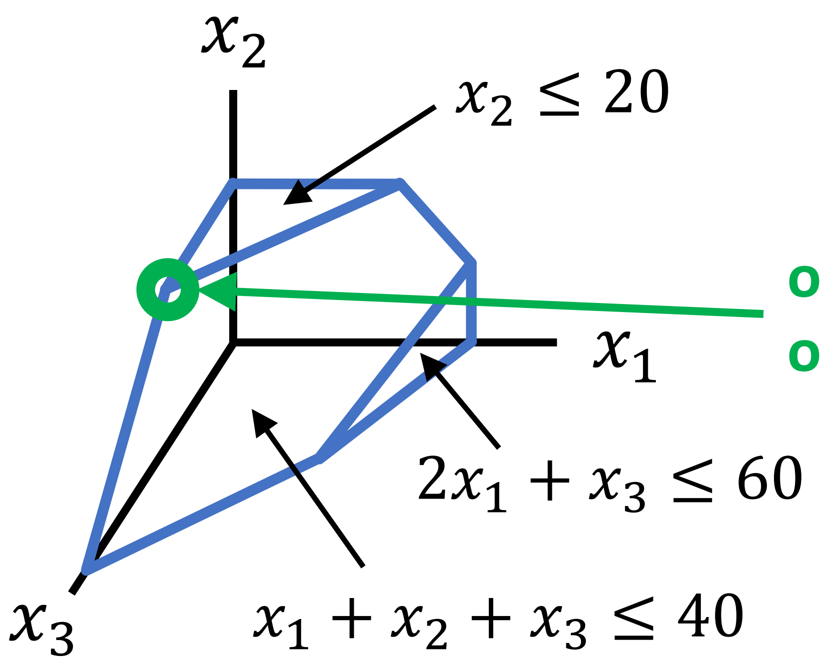
Subject to:

$$x_2 \le 20$$
$$x_1 + x_2 + x_3 \le 40$$
$$2x_1 + x_3 \le 60$$
$$x_1, x_2, x_3 \ge 0$$

Optimal Solution?

$$x_2 \le 20 \Longrightarrow 150x_2 \le 3000$$
$$x_1 + x_2 + x_3 \le 40$$

Example



$\mathbf{opt} = (\mathbf{0}, \mathbf{20}, \mathbf{20})$
$\mathbf{obj} = \mathbf{9000}$

$x_1, x_2, x_3 \in \mathbb{R}$

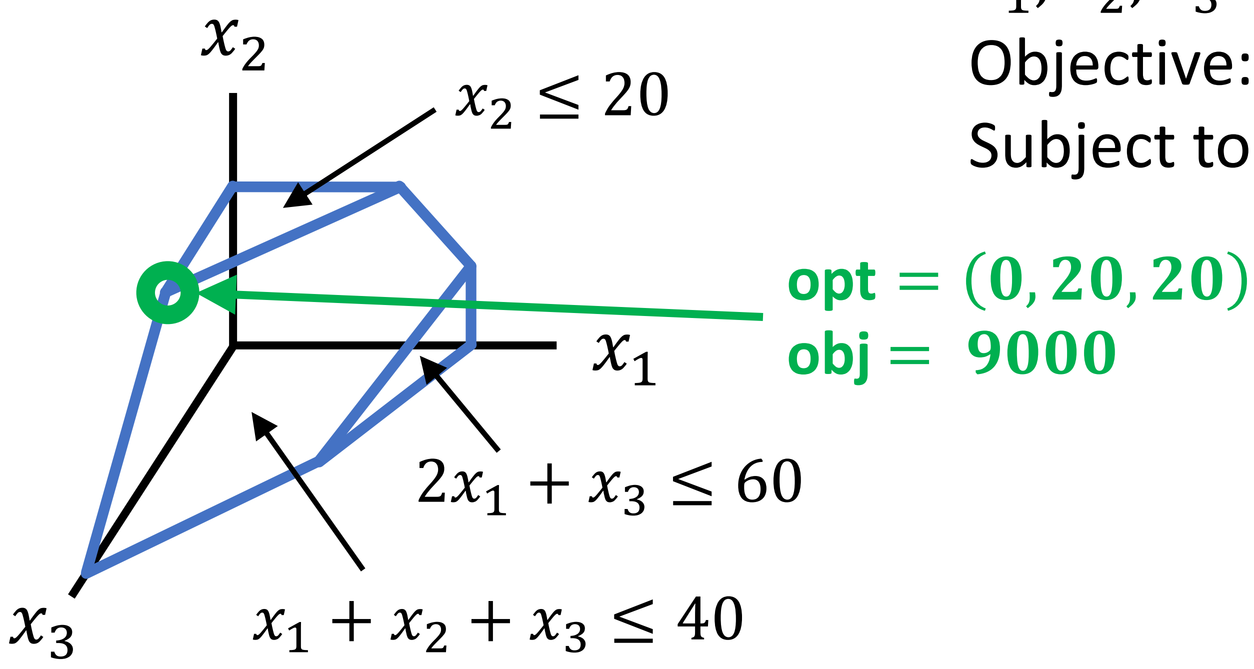
Objective: $\max 100x_1 + 300x_2 + 150x_3$

Subject to:
$x_2 \leq 20$
$x_1 + x_2 + x_3 \leq 40$
$2x_1 + x_3 \leq 60$
$x_1, x_2, x_3 \geq 0$

Optimal Solution?

$$x_2 \leq 20 \Longrightarrow 150x_2 \leq 3000$$
$$x_1 + x_2 + x_3 \leq 40 \Longrightarrow 150x_1 + 150x_2 + 150x_3 \leq 6000$$

Example



$x_1, x_2, x_3 \in \mathbb{R}$

Objective: $\max 100x_1 + 300x_2 + 150x_3$

Subject to: $x_2 \leq 20$

$x_1 + x_2 + x_3 \leq 40$

$2x_1 + x_3 \leq 60$

$x_1, x_2, x_3 \geq 0$

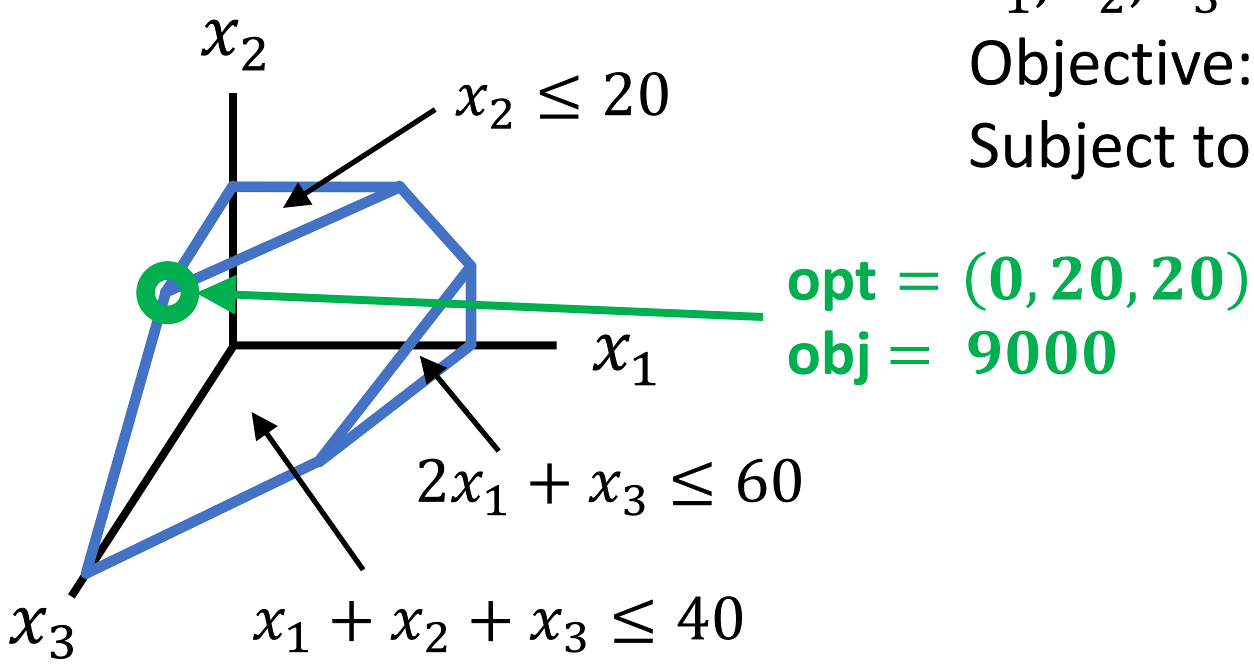
Optimal Solution?

$x_2 \leq 20 \Longrightarrow 150x_2 \leq 3000$

$x_1 + x_2 + x_3 \leq 40 \Longrightarrow 150x_1 + 150x_2 + 150x_3 \leq 6000$

$\Longrightarrow \quad 150x_1 + 300x_2 + 150x_3 \leq 9000$

Example



$x_1, x_2, x_3 \in \mathbb{R}$

Objective: $\max 100x_1 + 300x_2 + 150x_3$

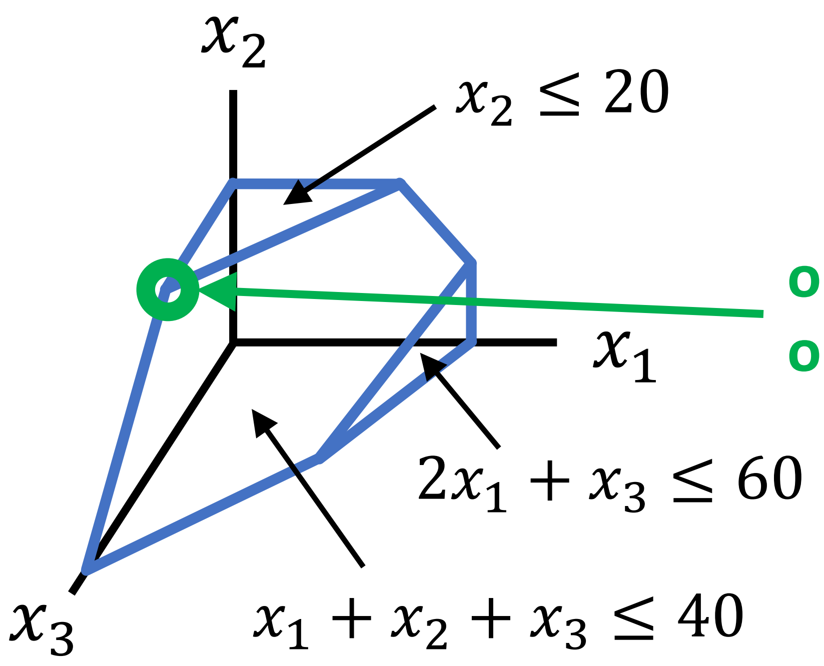
Subject to:
$x_2 \leq 20$
$x_1 + x_2 + x_3 \leq 40$
$2x_1 + x_3 \leq 60$
$x_1, x_2, x_3 \geq 0$

Optimal Solution?

$$x_2 \leq 20 \Longrightarrow 150x_2 \leq 3000$$
$$x_1 + x_2 + x_3 \leq 40 \Longrightarrow 150x_1 + 150x_2 + 150x_3 \leq 6000$$

$$\Longrightarrow 100x_1 + 300x_2 + 150x_3 \leq 150x_1 + 300x_2 + 150x_3 \leq 9000$$

Example



$\mathbf{opt} = (0, 20, 20)$
$\mathbf{obj} = 9000$

$x_1, x_2, x_3 \in \mathbb{R}$

Objective: $\max\ 100x_1 + 300x_2 + 150x_3$

Subject to: $x_2 \leq 20$

$x_1 + x_2 + x_3 \leq 40$

$2x_1 + x_3 \leq 60$

$x_1, x_2, x_3 \geq 0$

Optimal Solution?

$$x_2 \leq 20 \Longrightarrow 150x_2 \leq 3000$$

$$x_1 + x_2 + x_3 \leq 40 \Longrightarrow 150x_1 + 150x_2 + 150x_3 \leq 6000$$

$$\Longrightarrow 100x_1 + 300x_2 + 150x_3 \leq 150x_1 + 300x_2 + 150x_3 \leq 9000$$

LP Standard Form

Objective: $\max 100x_1 + 300x_2$
Subject to: $x_1 \leq 30$
$x_2 \leq 20$
$x_1 + x_2 \leq 40$
$x_1, x_2 \geq 0$

Objective: $\max c^T x$
Subject to: $A x \leq b$
$x \geq 0$

LP Standard Form

Objective: max $c^T x$
Subject to: $A x \leq b$
$x \geq 0$

Objective: $\max 100x_1 + 300x_2 + 150x_3$
Subject to:
$x_2 \leq 20$
$x_1 + x_2 + x_3 \leq 40$
$2x_1 + x_3 \leq 60$
$x_1, x_2, x_3 \geq 0$

Objective: $\max \begin{bmatrix} 100 & 300 & 150 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Subject to: $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \leq \begin{bmatrix} 20 \\ 40 \\ 60 \end{bmatrix}$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

LP Standard Form

Objective: $\max 100x_1 + 300x_2$
Subject to: $x_1 \le 30$
$x_2 \le 20$
$x_1 + x_2 \le 40$
$x_1, x_2 \ge 0$

$\Rightarrow$

$\begin{pmatrix}100\\300\end{pmatrix}$ $\begin{pmatrix}x_1\\x_2\end{pmatrix}$

Objective: $\max c^T x$
Subject to: $A x \le b$
$x \ge 0$

$$\begin{pmatrix}1 & 0\\0 & 1\\1 & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} \le \begin{pmatrix}30\\20\\40\end{pmatrix}$$

Every LP can be turned into standard form.

1.

2.

3.

4.

LP Standard Form

Objective: $\max 100x_1 + 300x_2$
Subject to: $x_1 \leq 30$
$x_2 \leq 20$
$x_1 + x_2 \leq 40$
$x_1, x_2 \geq 0$

$\begin{pmatrix} 100 \\ 300 \end{pmatrix}$ $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Objective: max $c^T x$
Subject to: $A x \leq b$
$x \geq 0$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 30 \\ 20 \\ 40 \end{pmatrix}$$

Every LP can be turned into standard form.

1. Minimization → Maximization: ?

2. $\geq$ Constraints → $\leq$:

3. Equality Constraints → $\leq$:

4. Unrestricted sign x_1 → $x_1 \geq 0$:

LP Standard Form

Objective: $\max 100x_1 + 300x_2$
Subject to: $x_1 \leq 30$
$x_2 \leq 20$
$x_1 + x_2 \leq 40$
$x_1, x_2 \geq 0$

$\begin{pmatrix} 100 \\ 300 \end{pmatrix}$ $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Objective: max c^T x
Subject to: A x ≤ b
x ≥ 0

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 30 \\ 20 \\ 40 \end{pmatrix}$$

Every LP can be turned into standard form.

1. Minimization → Maximization: Multiply objective coefficients by -1.
$$\min \alpha x_1 + \beta x_2 \rightarrow \max -\alpha x_1 - \beta x_2$$
2. ≥ Constraints → ≤: ?
3. Equality Constraints → ≤:
4. Unrestricted sign $x_1 \rightarrow x_1 \geq 0$:

LP Standard Form

Objective: $\max 100x_1 + 300x_2$
Subject to: $x_1 \leq 30$
$x_2 \leq 20$
$x_1 + x_2 \leq 40$
$x_1, x_2 \geq 0$

$\begin{pmatrix} 100 \\ 300 \end{pmatrix}$ $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Objective: max c^T x
Subject to: A x ≤ b
x ≥ 0

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 30 \\ 20 \\ 40 \end{pmatrix}$$

Every LP can be turned into standard form.

1. Minimization → Maximization: Multiply objective coefficients by -1.
   $$\min \alpha x_1 + \beta x_2 \rightarrow \max -\alpha x_1 - \beta x_2$$
2. ≥ Constraints → ≤: Negate inequality.
   $$x_1 + x_2 \geq \alpha \rightarrow -x_1 - x_2 \leq -\alpha$$
3. Equality Constraints → ≤: ?
4. Unrestricted sign $x_1 \rightarrow x_1 \geq 0$:

LP Standard Form

Objective: $\max 100x_1 + 300x_2$
Subject to: $x_1 \le 30$
$x_2 \le 20$
$x_1 + x_2 \le 40$
$x_1, x_2 \ge 0$

$\begin{pmatrix} 100 \\ 300 \end{pmatrix}$ $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Objective: max c^T x
Subject to: A x ≤ b
x ≥ 0

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \le \begin{pmatrix} 30 \\ 20 \\ 40 \end{pmatrix}$$

Every LP can be turned into standard form.

1. Minimization → Maximization: Multiply objective coefficients by -1.
   $$\min \alpha x_1 + \beta x_2 \rightarrow \max -\alpha x_1 - \beta x_2$$
2. $\ge$ Constraints → $\le$: Negate inequality.
   $$x_1 + x_2 \ge \alpha \rightarrow -x_1 - x_2 \le -\alpha$$
3. Equality Constraints → $\le$: Introduce $\ge$ and $\le$ constraints.
   $$x_1 + x_2 = \alpha \rightarrow x_1 + x_2 \ge \alpha \text{ and } x_1 + x_2 \le \alpha$$
4. Unrestricted sign $x_1 \rightarrow x_1 \ge 0$: ?

LP Standard Form

Objective: $\max 100x_1 + 300x_2$
Subject to: $x_1 \le 30$
$x_2 \le 20$
$x_1 + x_2 \le 40$
$x_1, x_2 \ge 0$

$$c = \begin{pmatrix} 100 \\ 300 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

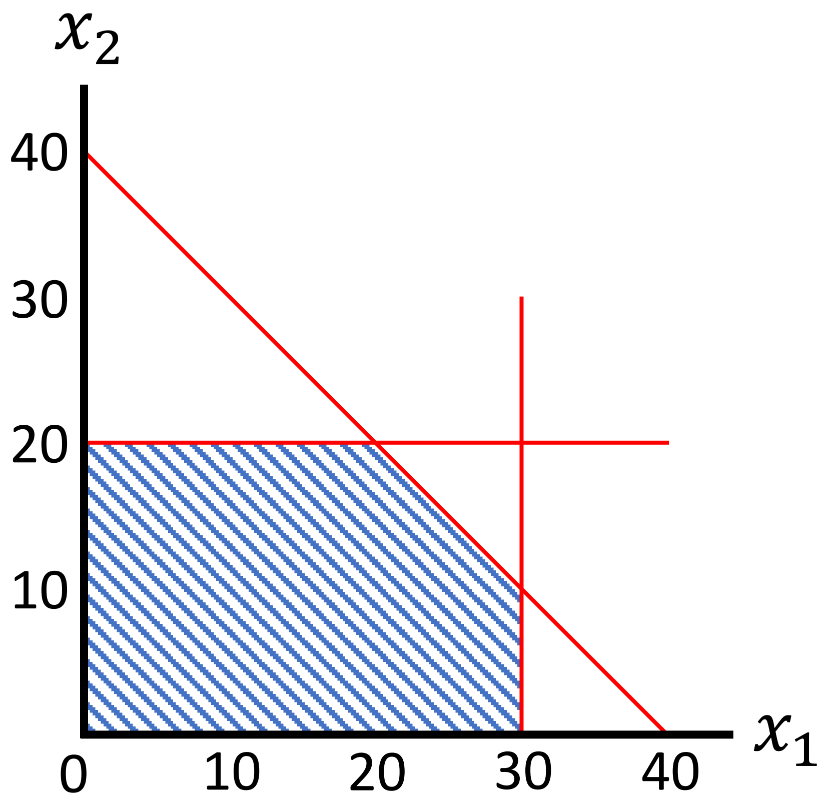
Objective: max c^T x
Subject to: A x ≤ b
x ≥ 0

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \le \begin{pmatrix} 30 \\ 20 \\ 40 \end{pmatrix}$$

Every LP can be turned into standard form.

1. Minimization → Maximization: Multiply objective coefficients by -1.
$$\min \alpha x_1 + \beta x_2 \rightarrow \max -\alpha x_1 - \beta x_2$$
2. ≥ Constraints → ≤: Negate inequality.
$$x_1 + x_2 \ge \alpha \rightarrow -x_1 - x_2 \le -\alpha$$
3. Equality Constraints → ≤: Introduce ≥ and ≤ constraints.
$$x_1 + x_2 = \alpha \rightarrow x_1 + x_2 \ge \alpha \text{ and } x_1 + x_2 \le \alpha$$
4. Unrestricted sign $x_1 \rightarrow x_1 \ge 0$: Replace x_1 with $x_1^+ - x_1^-$ (for $x_1^+ \ge 0$, $x_1^- \ge 0$).
$$x_1 + x_2 \le \alpha \rightarrow x_1^+ - x_1^- + x_2 \le \alpha$$

Optimal Value



Objective: $\max f(x_1, x_2)$

Subject to: $c_1(x_1, x_2)$

$c_2(x_1, x_2)$

$\vdots$

$c_n(x_1, x_2)$

Properties of optimal solutions:

1. Optimal value occurs at a vertex.
2. Local optimum is global optimum.

Algorithm to find optimal solution:

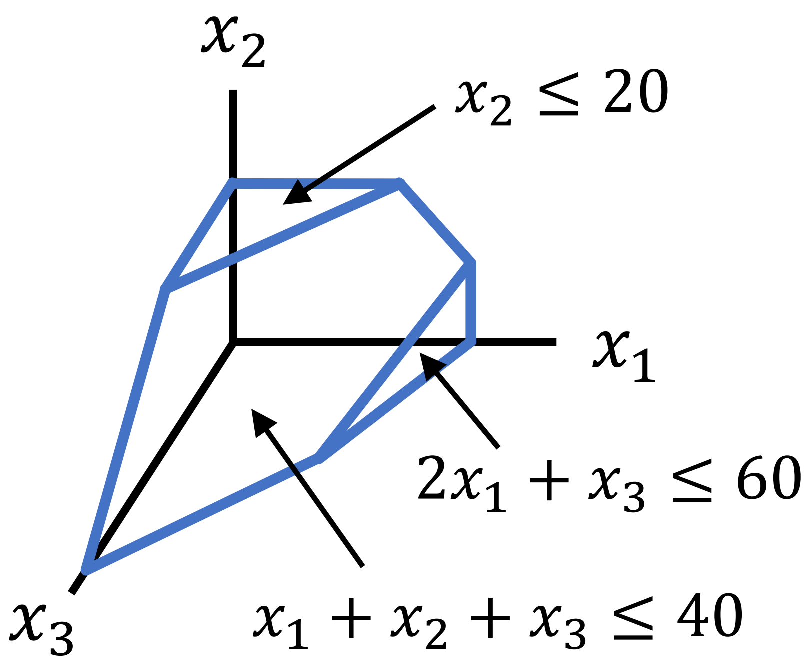
Test each vertex in order until no neighbors have larger (or smaller) value.

Simplex Algorithm

```
Simplex(LP)
   v = vertex in feasible region of LP
   while ∃ neighbor v' with better objective value
      v = v'
   return v
```

How do we find vertices?

Example

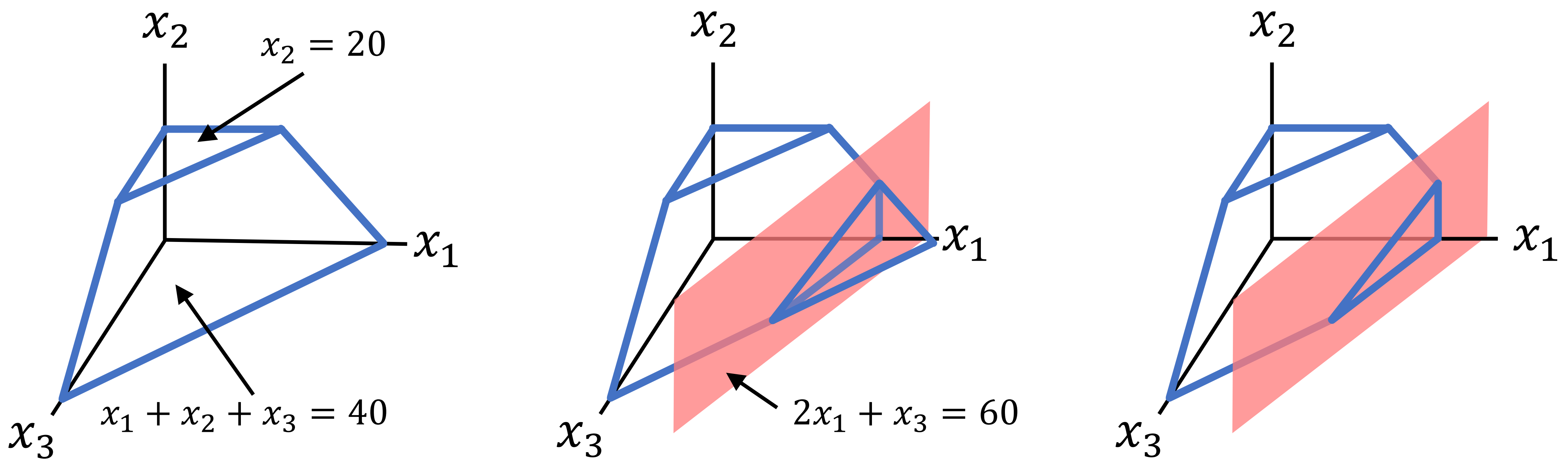


$x_1, x_2, x_3 \in \mathbb{R}$

Objective: $\max 100x_1 + 300x_2 + 150x_3$

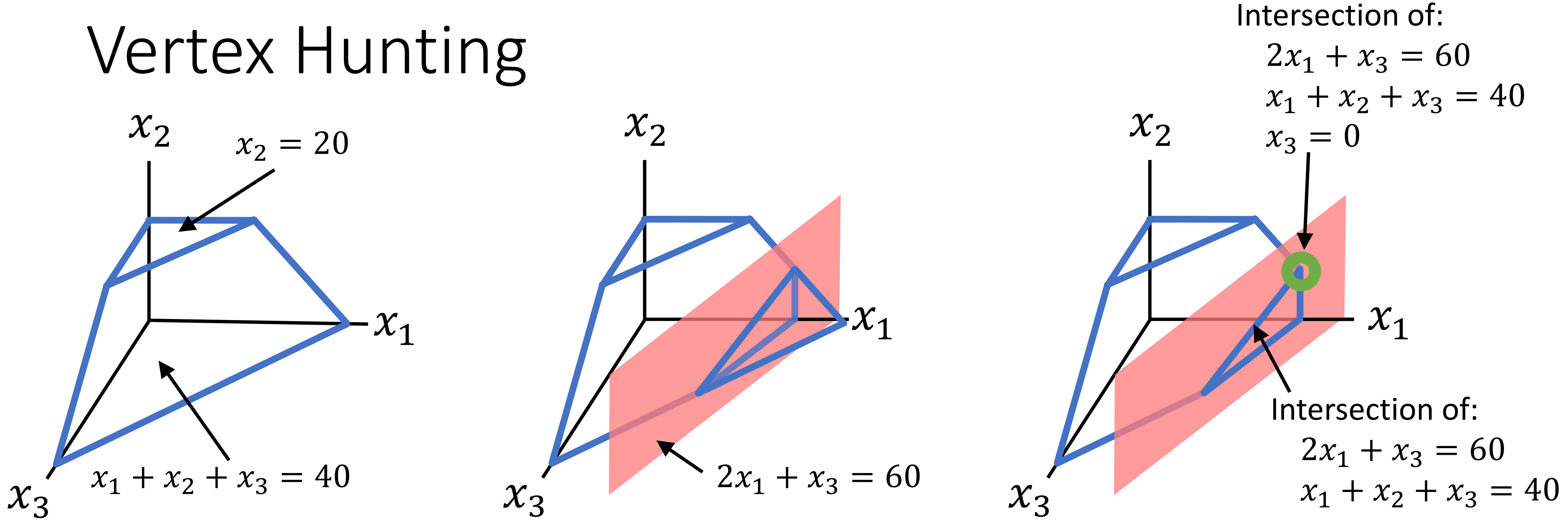
Subject to:
$$x_2 \le 20$$
$$x_1 + x_2 + x_3 \le 40$$
$$2x_1 + x_3 \le 60$$
$$x_1, x_2, x_3 \ge 0$$

Vertex Hunting



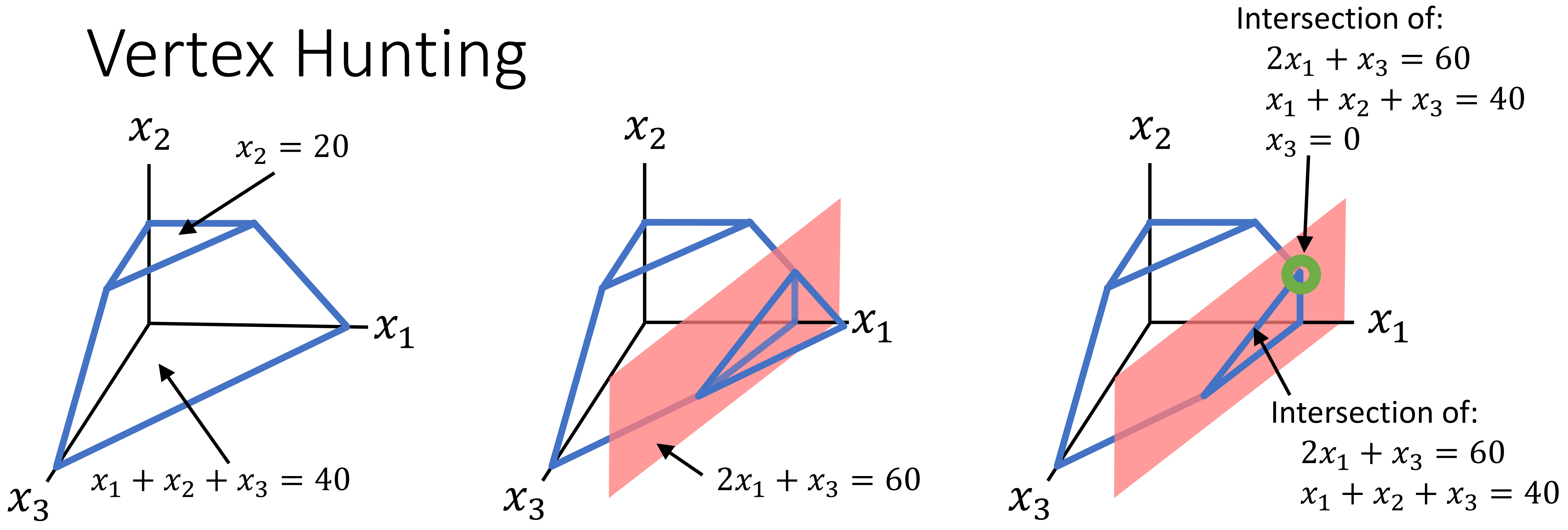
Feasible region construction: Area contained within intersection of hyperplanes represented by inequality constraints.

Vertex Hunting



Definition: A *vertex* is a point in $\mathbb{R}^n$ that uniquely satisfies some inequalities with equality, and is feasible.

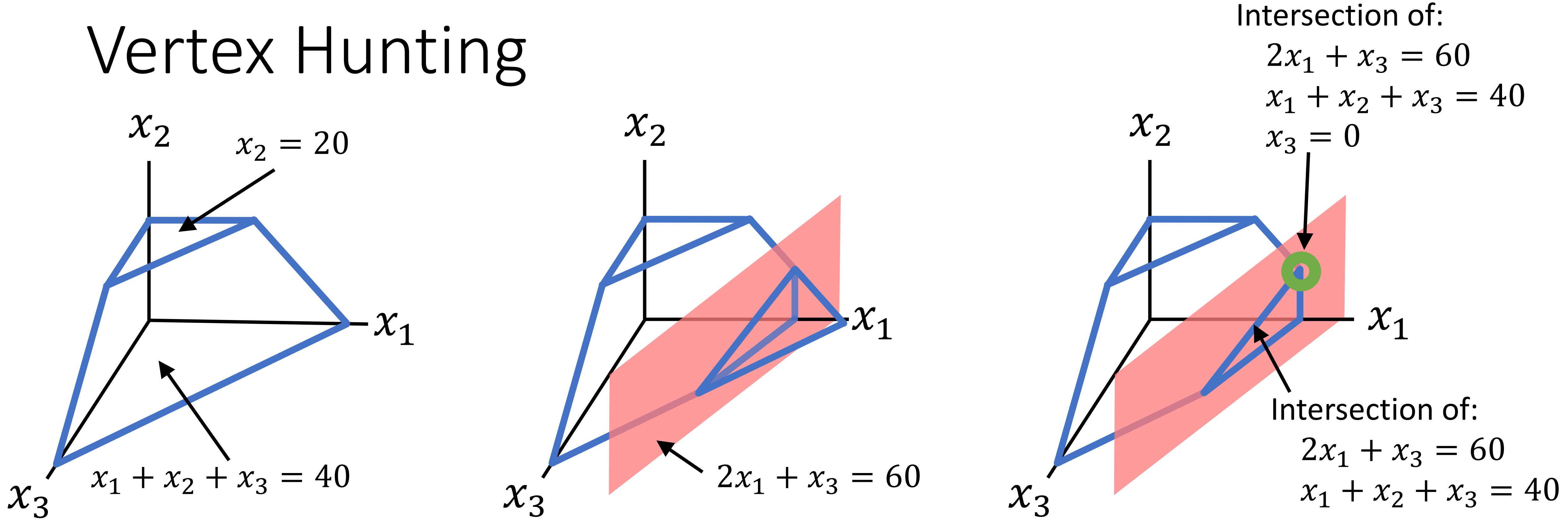
Vertex Hunting



Definition: A *vertex* is a point in $\mathbb{R}^n$ that uniquely satisfies some inequalities with equality, and is feasible.

How many (linearly independent) inequalities define a vertex?

Vertex Hunting

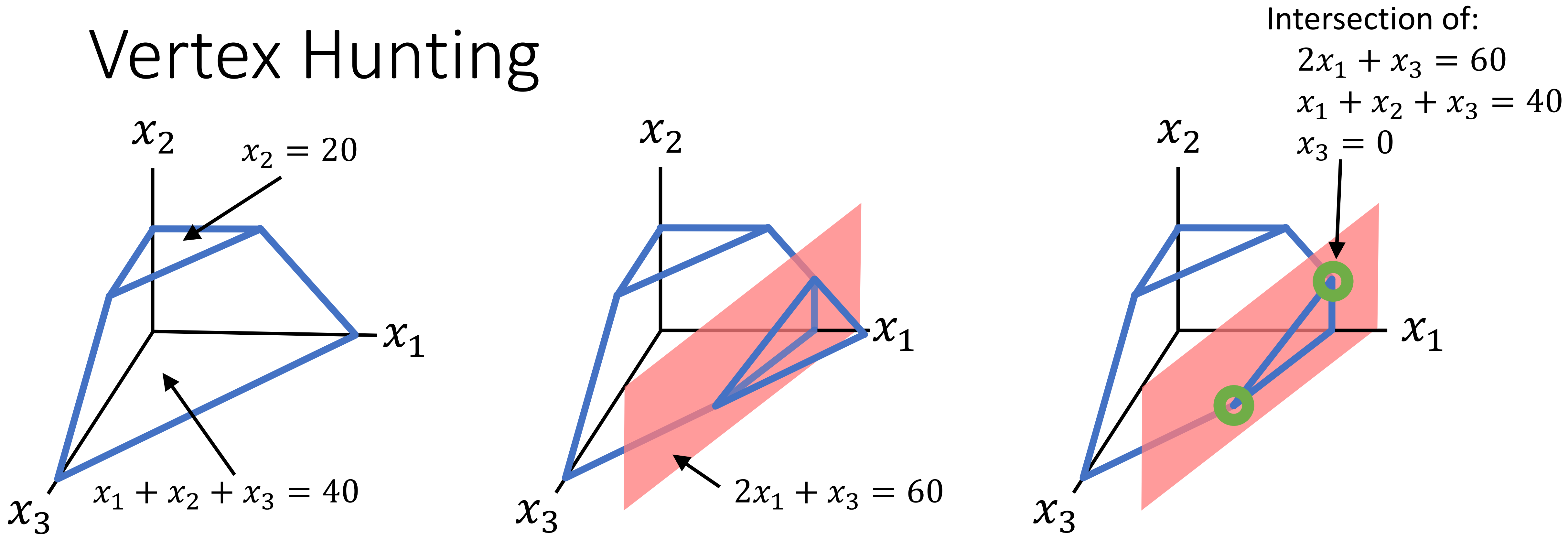


Definition: A *vertex* is a point in $\mathbb{R}^n$ that uniquely satisfies some inequalities with equality, and is feasible.

How many (linearly independent) inequalities define a vertex?

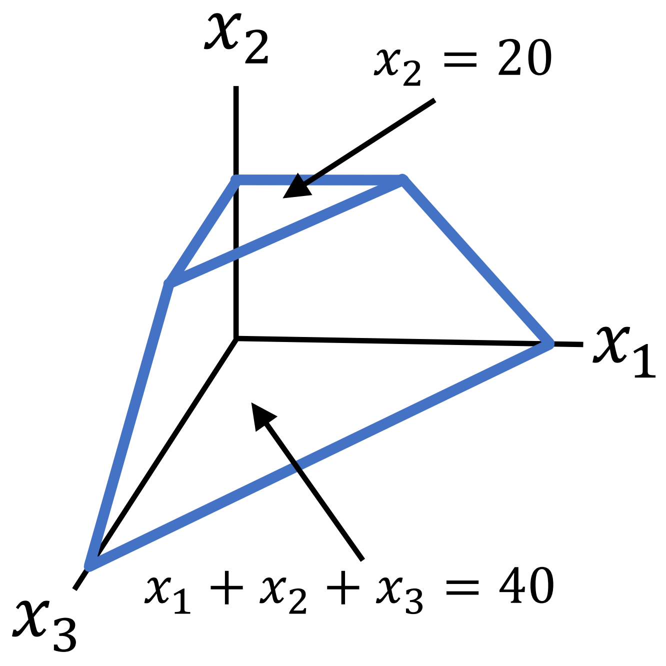
n.

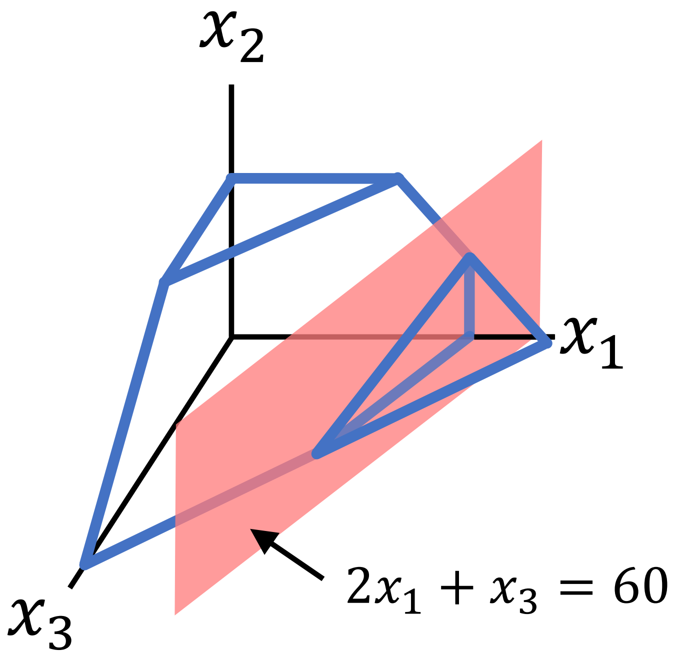
Vertex Hunting

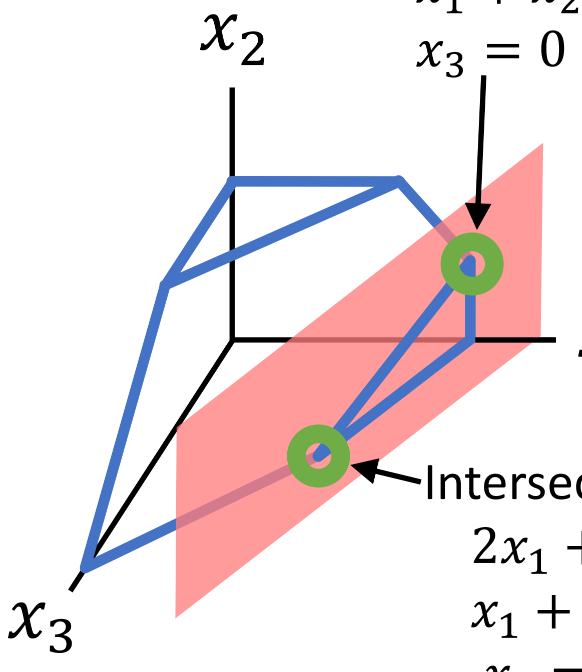


Definition: Two vertices are *neighbors* if...?

Vertex Hunting

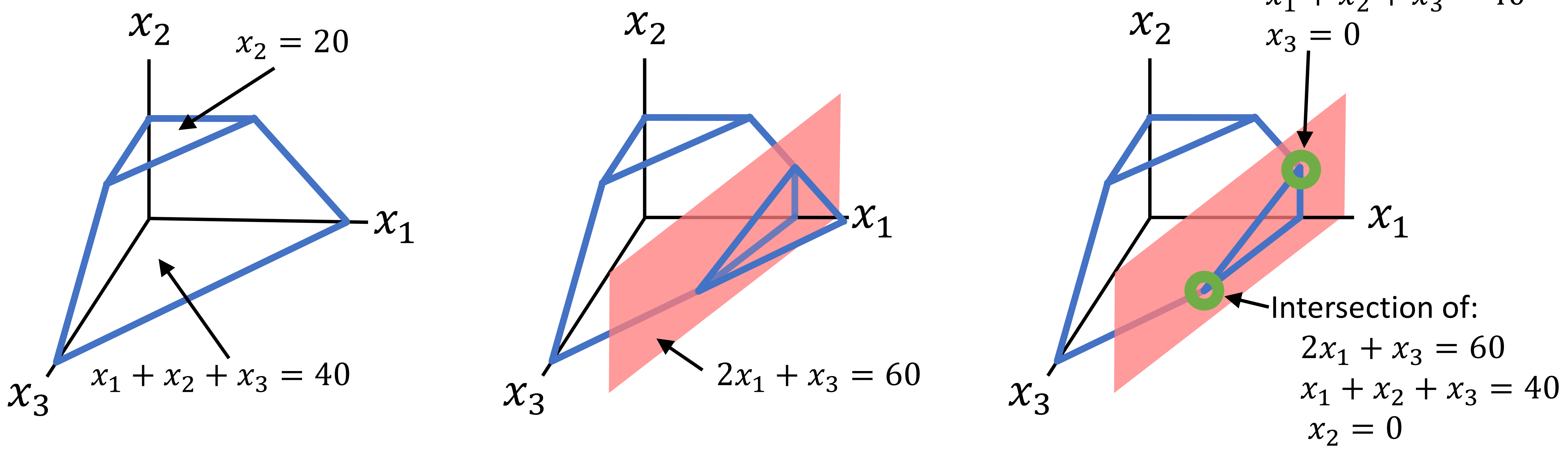






Definition: Two vertices are *neighbors* if they share $n - 1$ defining inequalities.

Vertex Hunting



Definition: Two vertices are *neighbors* if they share $n - 1$ defining inequalities.

Plan: Move from vertex to vertex by following line formed by intersection of $n - 1$ inequalities.