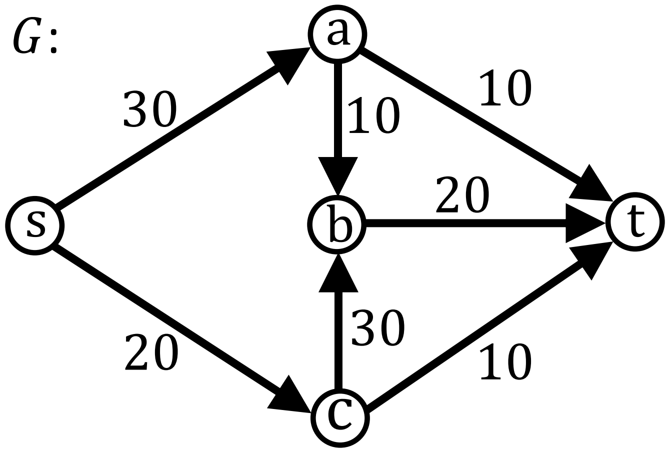
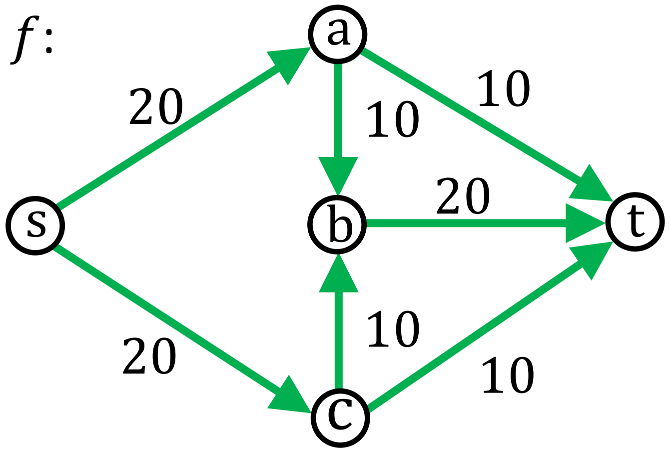
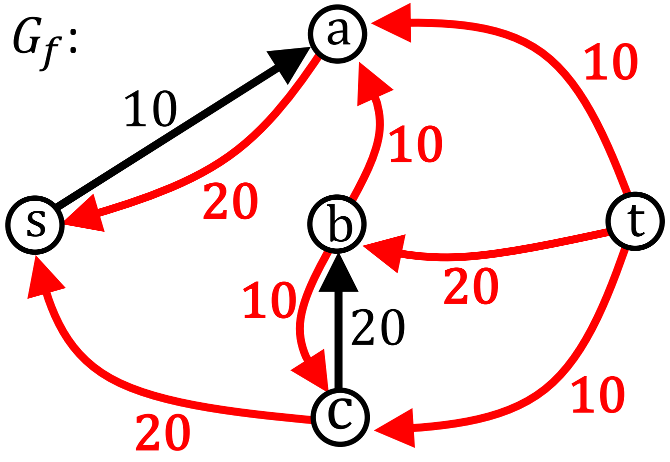
Flow Networks
CSCI 532

Ford-Fulkerson








```
Max-Flow(G)
  f(e) = 0 for all
  while s-t path in
    P = simple s-t
    f'= augment(f,
    f = f'
    Gf = Gf'
  return f
```

```
augment(f, P)
  bottleneck(P,f)
  each edge (u, v) in P
  f (u, v) is a back edge
    f((v, u)) -= b
  se
    f((u, v)) += b
  return f
```

Need to show:

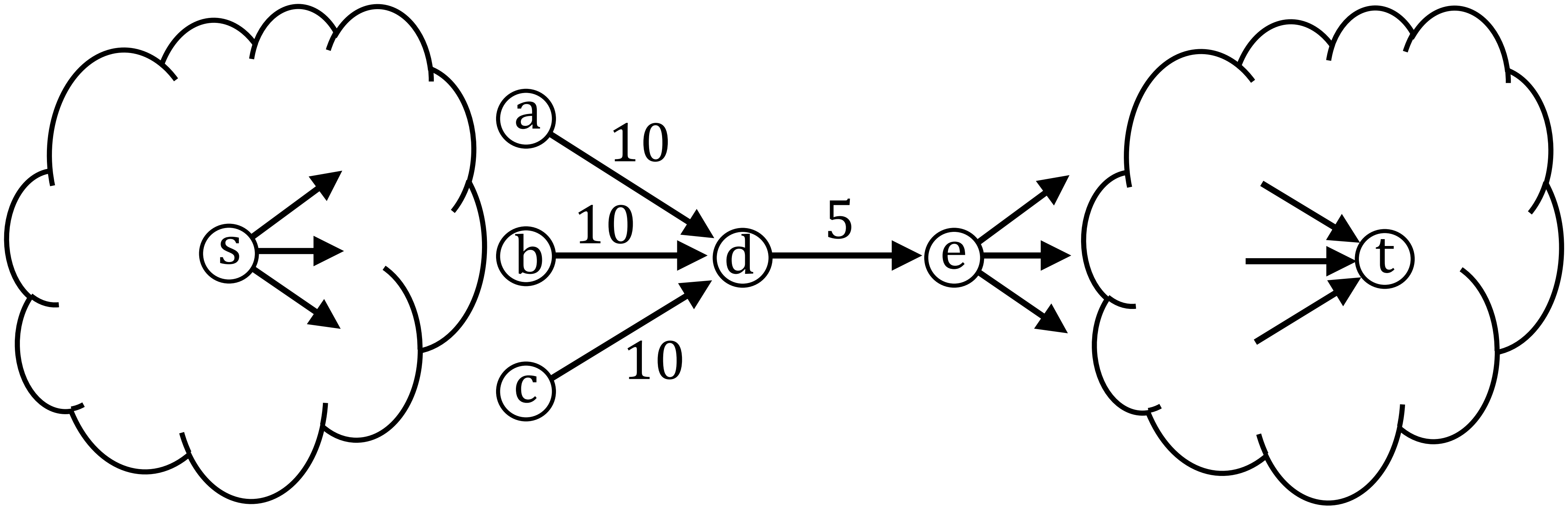
1. ~~Validity.~~
2. ~~Running time.~~
3. Finds max flow.

Optimality

Theorem: The flow returned by the Ford-Fulkerson algorithm is a maximum flow.

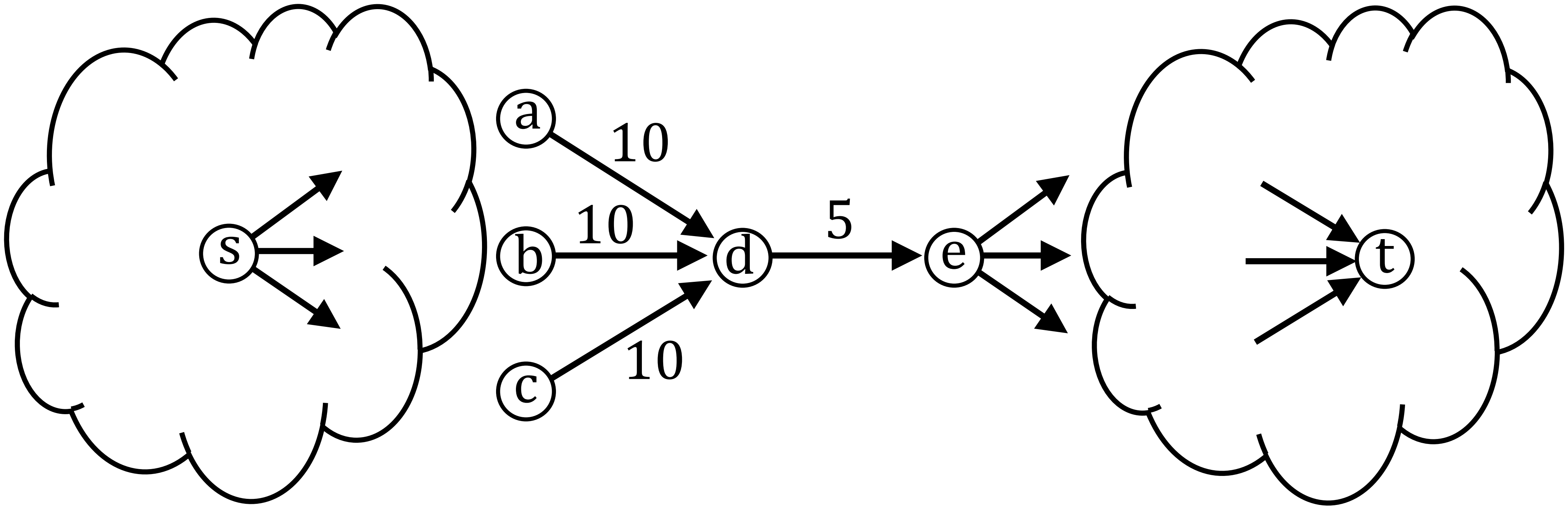
Proof: ...

Optimality



What can we say about the maximum flow on this network?

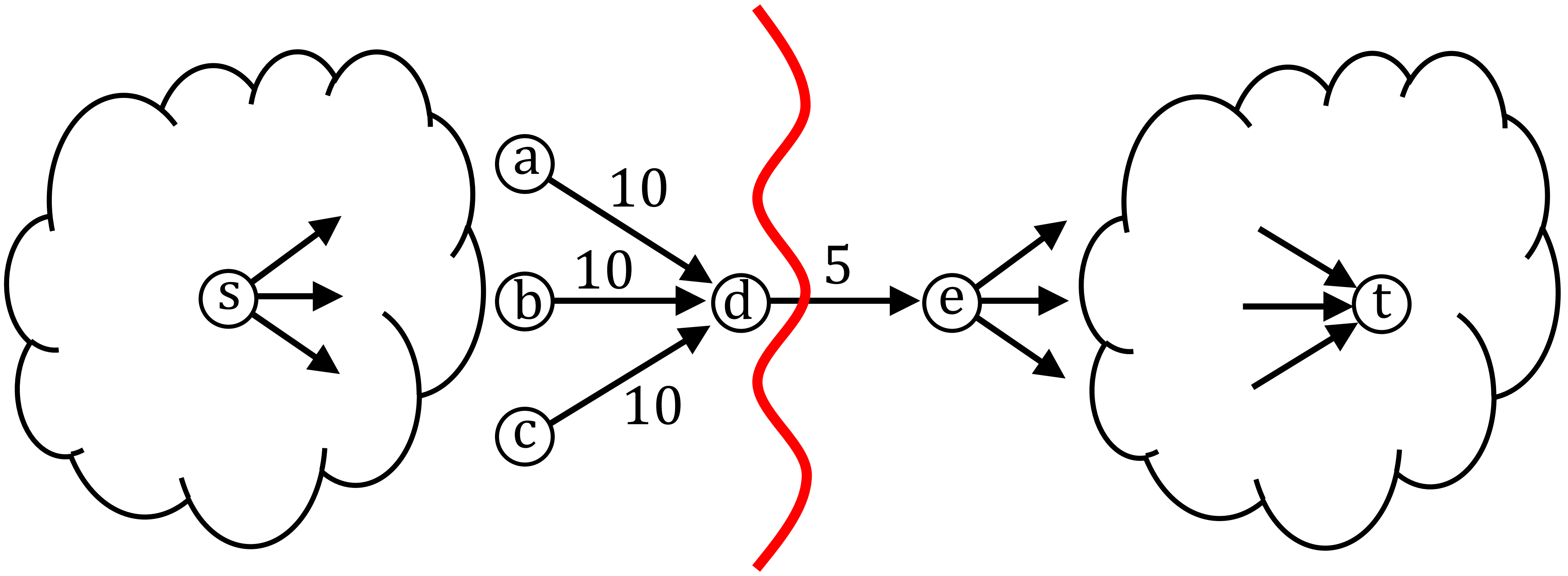
Optimality



What can we say about the maximum flow on this network?

It's not larger than 5.

Optimality



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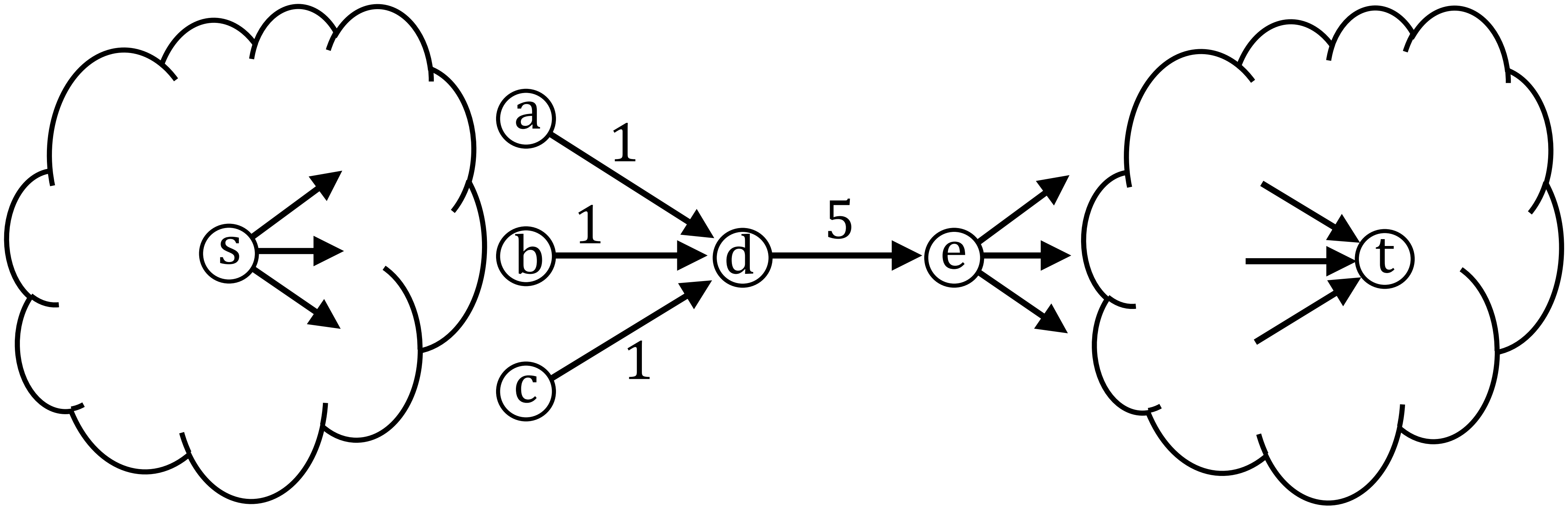

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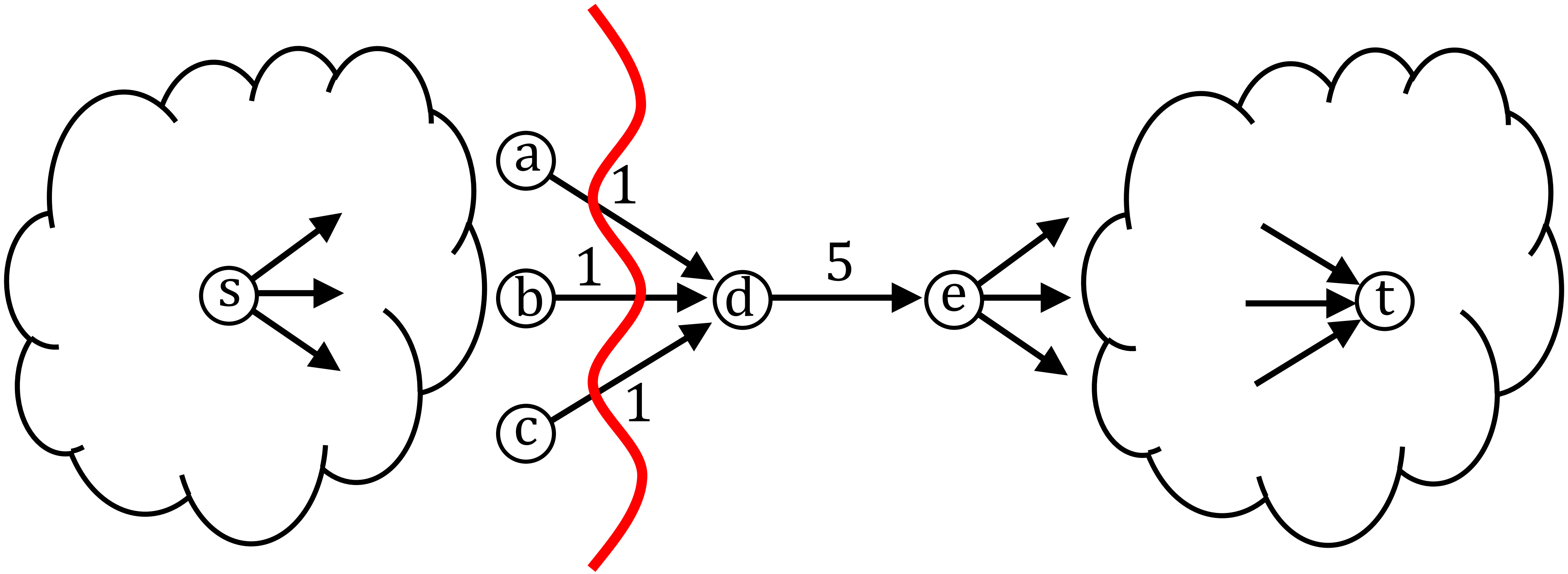
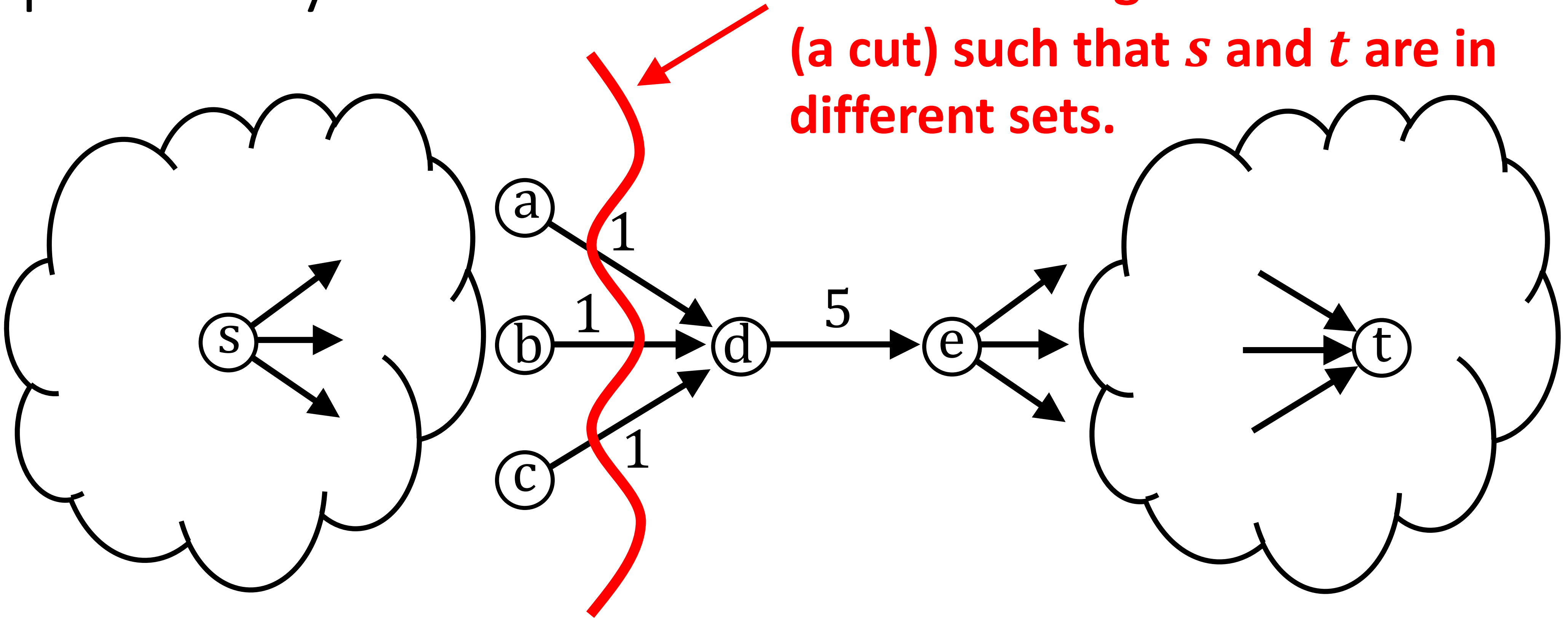


What can we say about the maximum flow on this network?

It’s not larger than 3.

Optimality

$s - t$ cut: Segmentation of vertices (a cut) such that s and t are in different sets.



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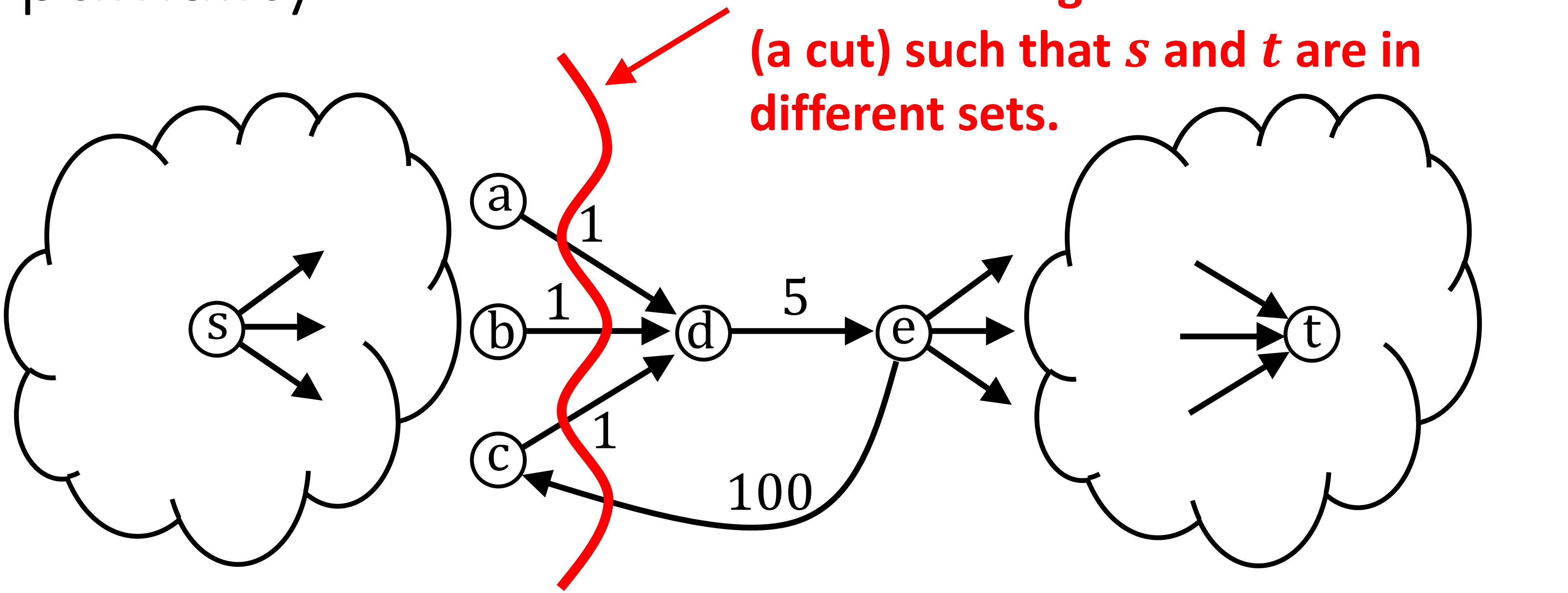

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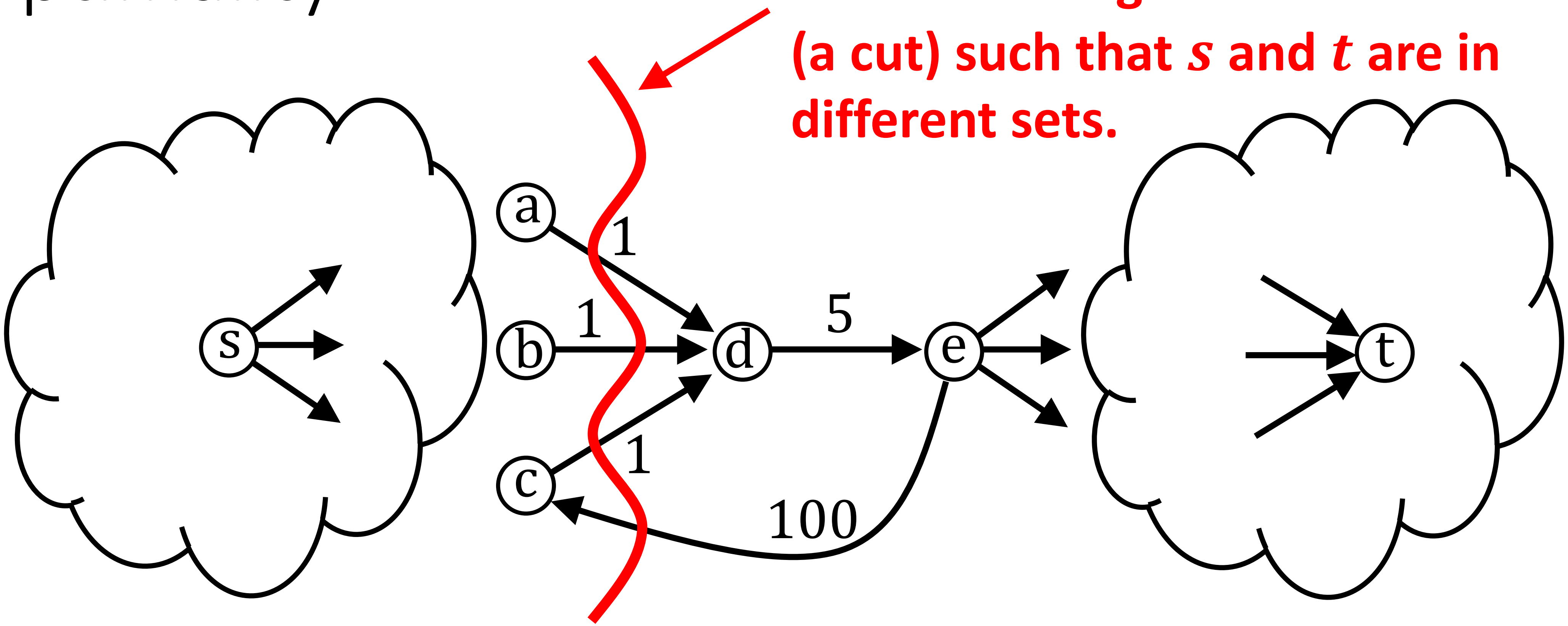

What can we say about the maximum flow on this network?

It's not larger than 3, since the net transfer from s to t is bounded by the bottleneck cut.

Optimality

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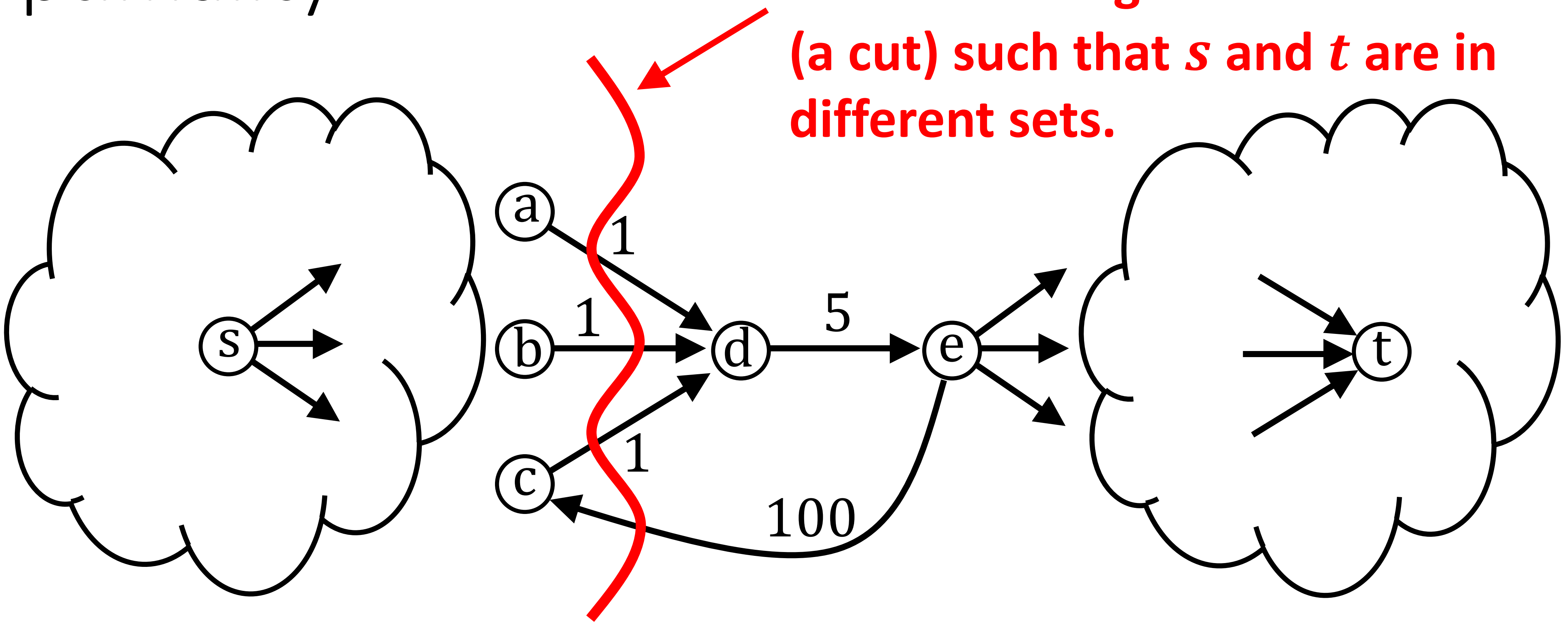

The capacity of a cut is the sum of the capacities leaving s's set.

Optimality

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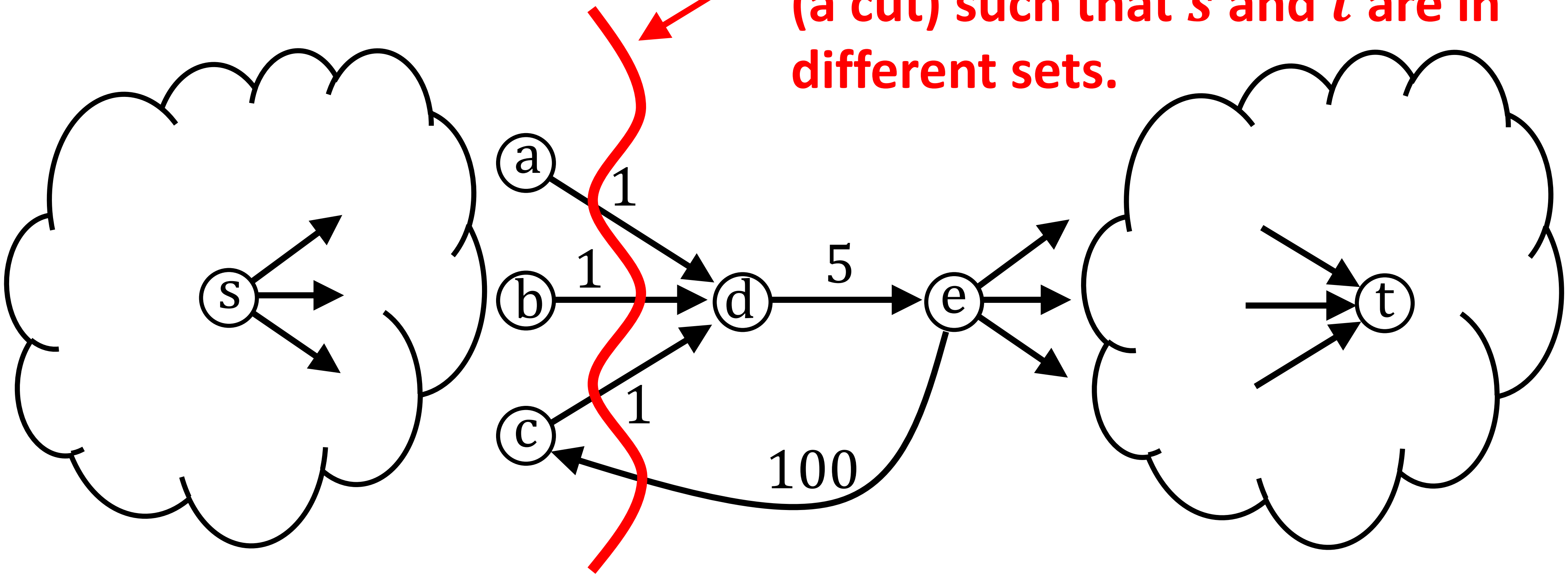

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Proposed Theorem?

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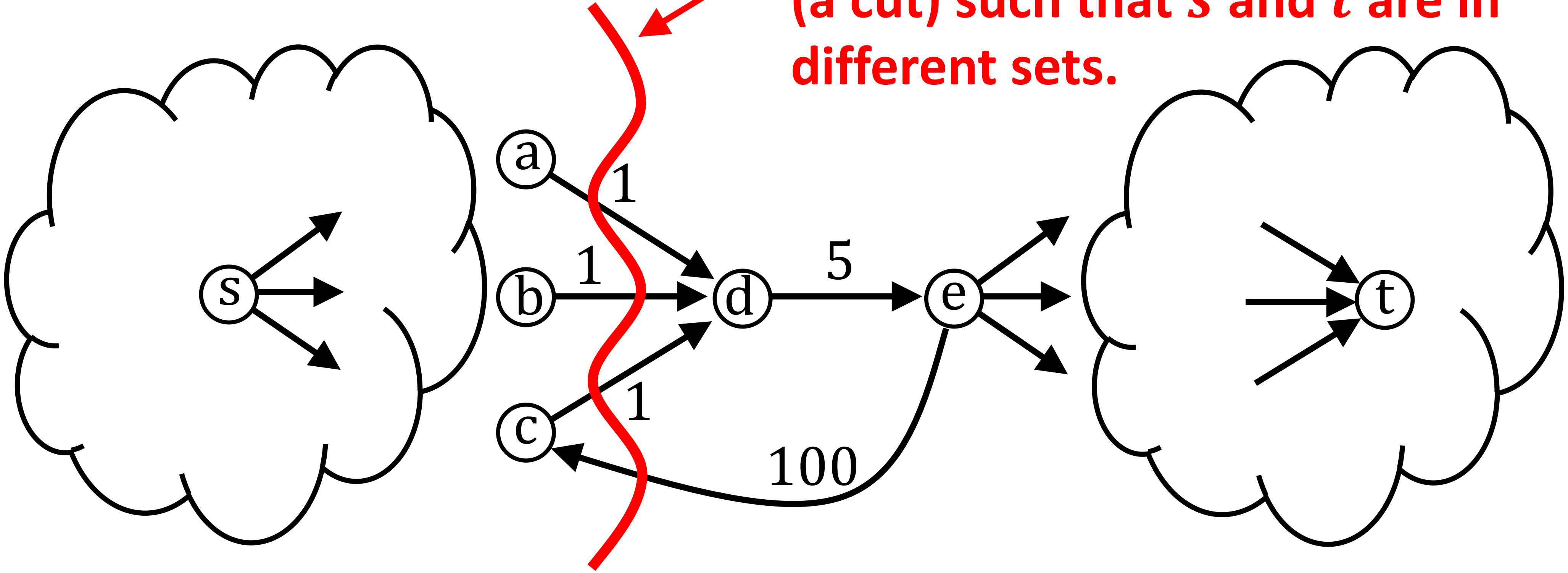


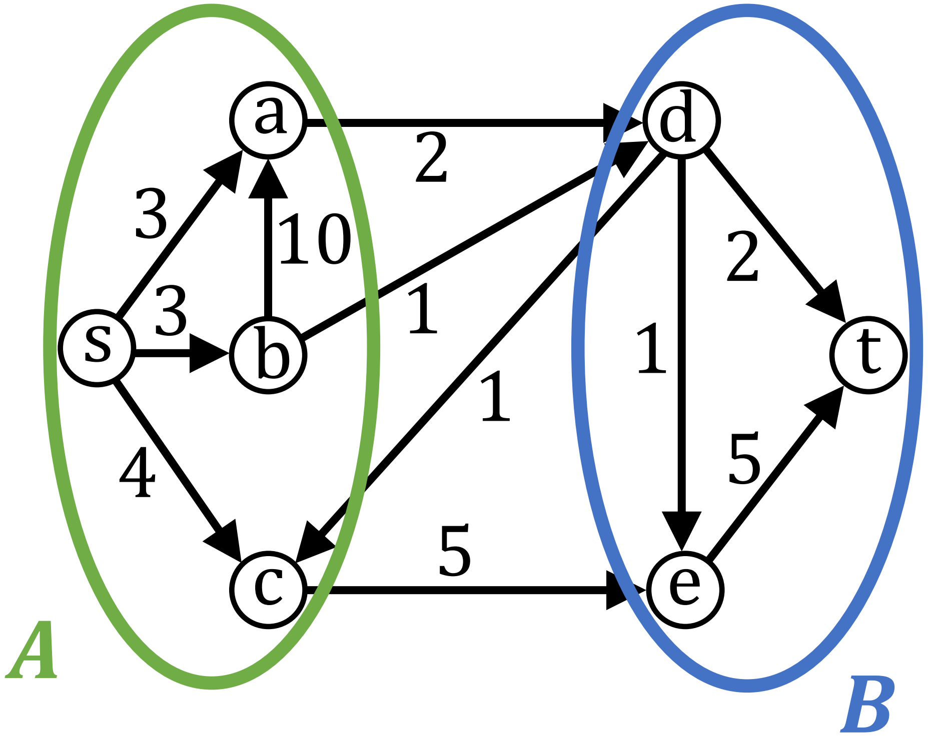
The capacity of a cut is the sum of the capacities leaving s's set.

Proposed Theorem?

The minimum capacity cut = the maximum flow.

$s - t$ cuts

Definitions: Suppose G is a flow network and nodes in G are divided into two sets, $\boldsymbol{A}$ and $\boldsymbol{B}$, such that $s \in A$ and $t \in B$. We call (A, B) an *$s - t$ cut*. The *capacity* of the cut, $c(A, B)$, is the sum of capacities of all edges out of A.



$c(A, B) = 8$

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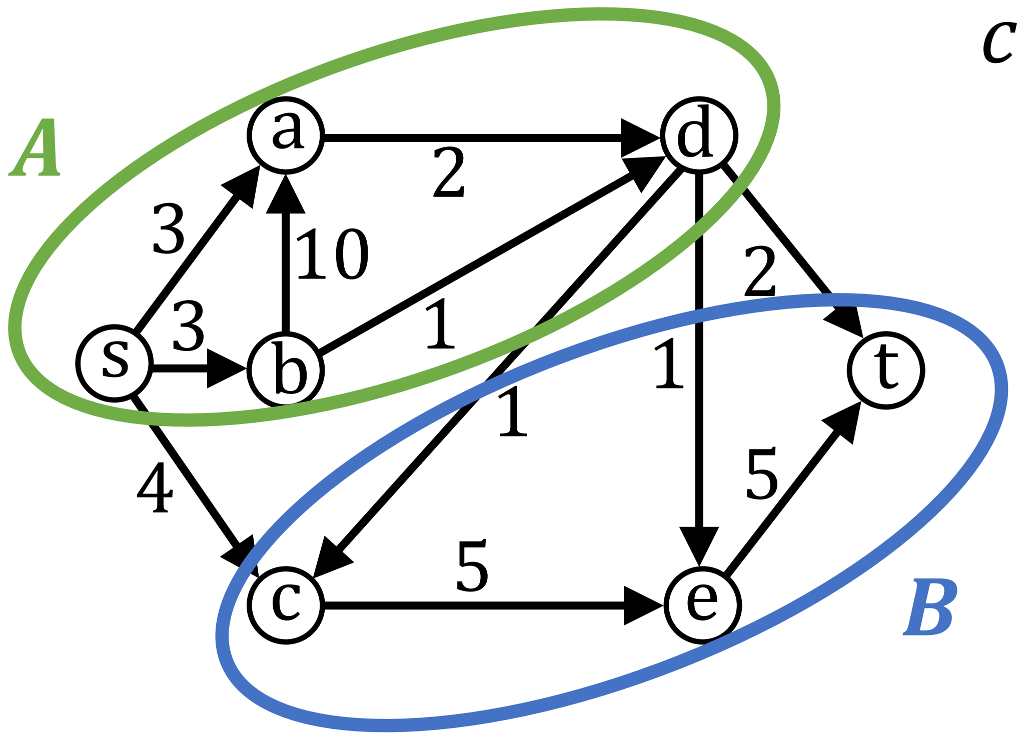

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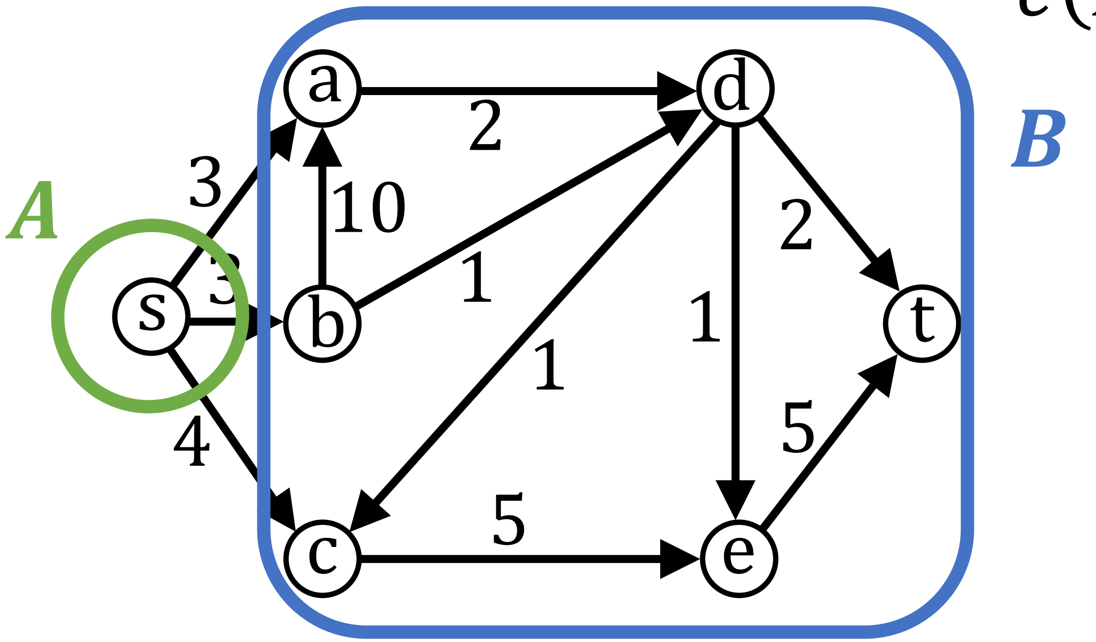
$$c(A, B) = 10$$



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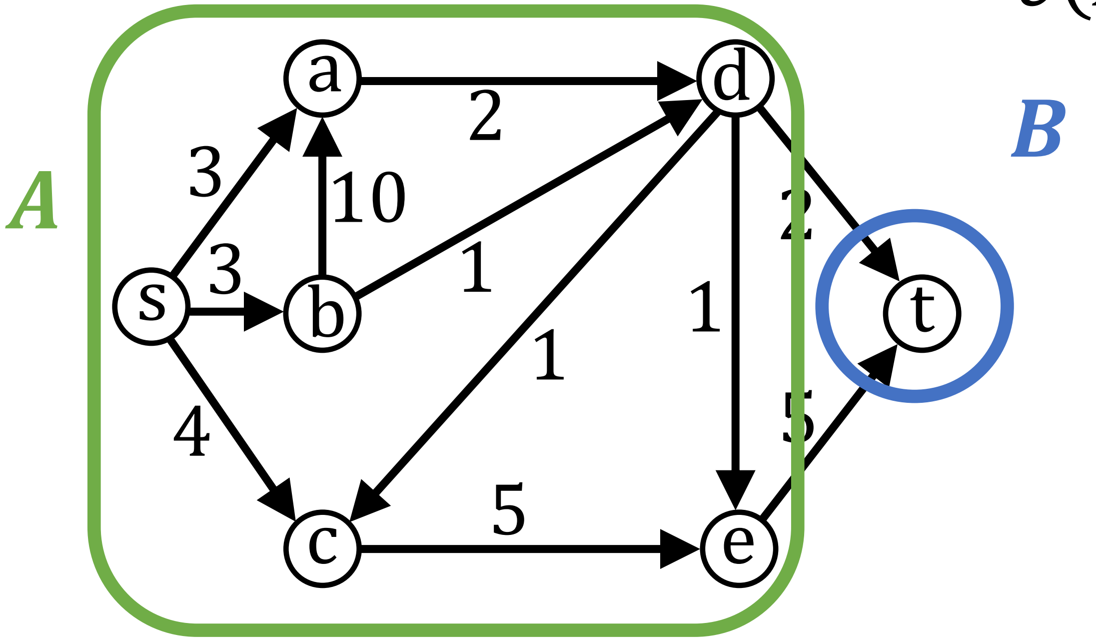
$$c(A, B) = 7$$



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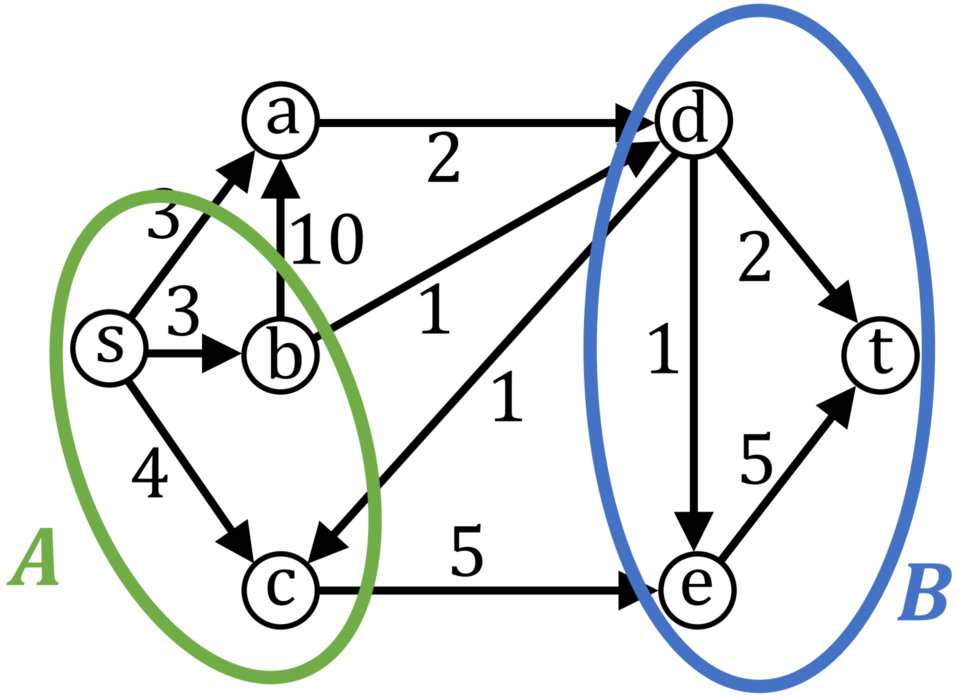

Invalid cut! Every vertex needs to be is in one of the sets!

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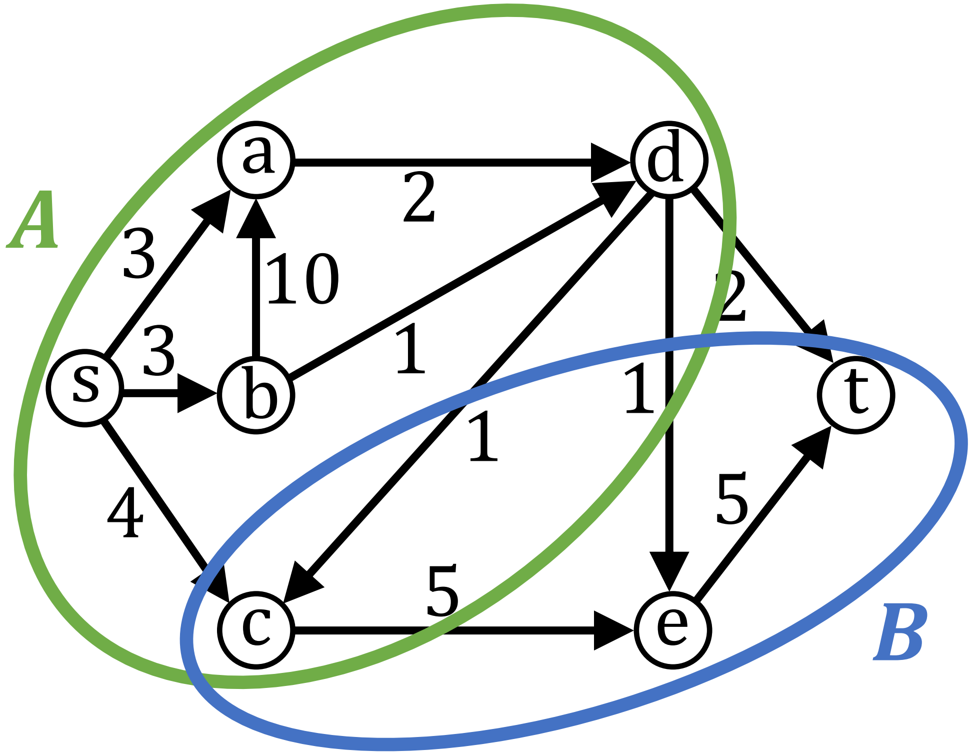

Invalid cut! Every vertex needs to be in <u>exactly</u> one of the sets!

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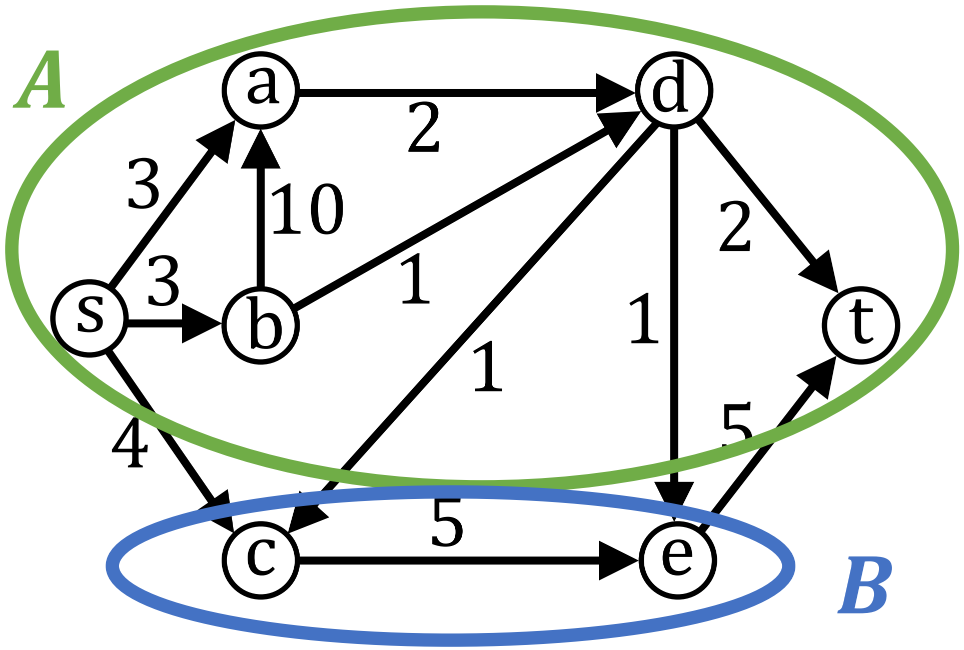
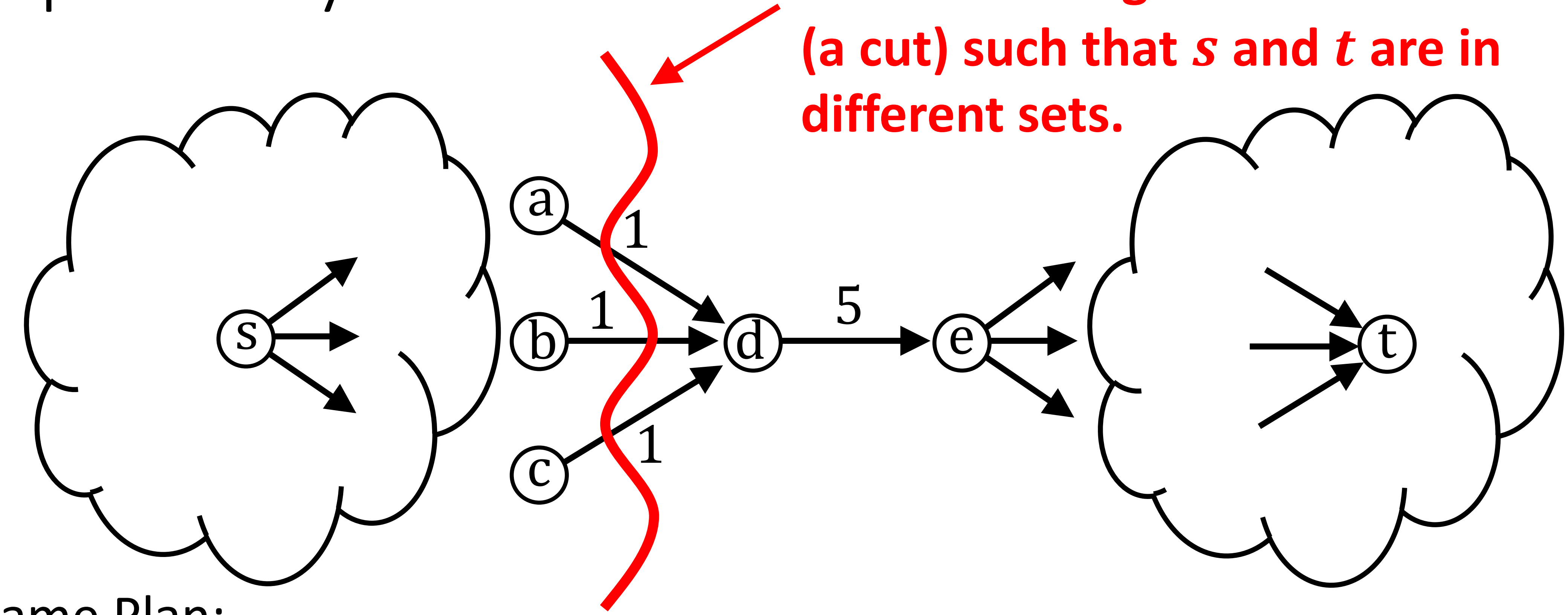


Invalid $s - t$ cut! s and t need to be in different sets!

Optimality



$s - t$ cut: Segmentation of vertices (a cut) such that s and t are in different sets.

Game Plan:

1. Show that value of every flow is $\leq$ capacity of every cut.

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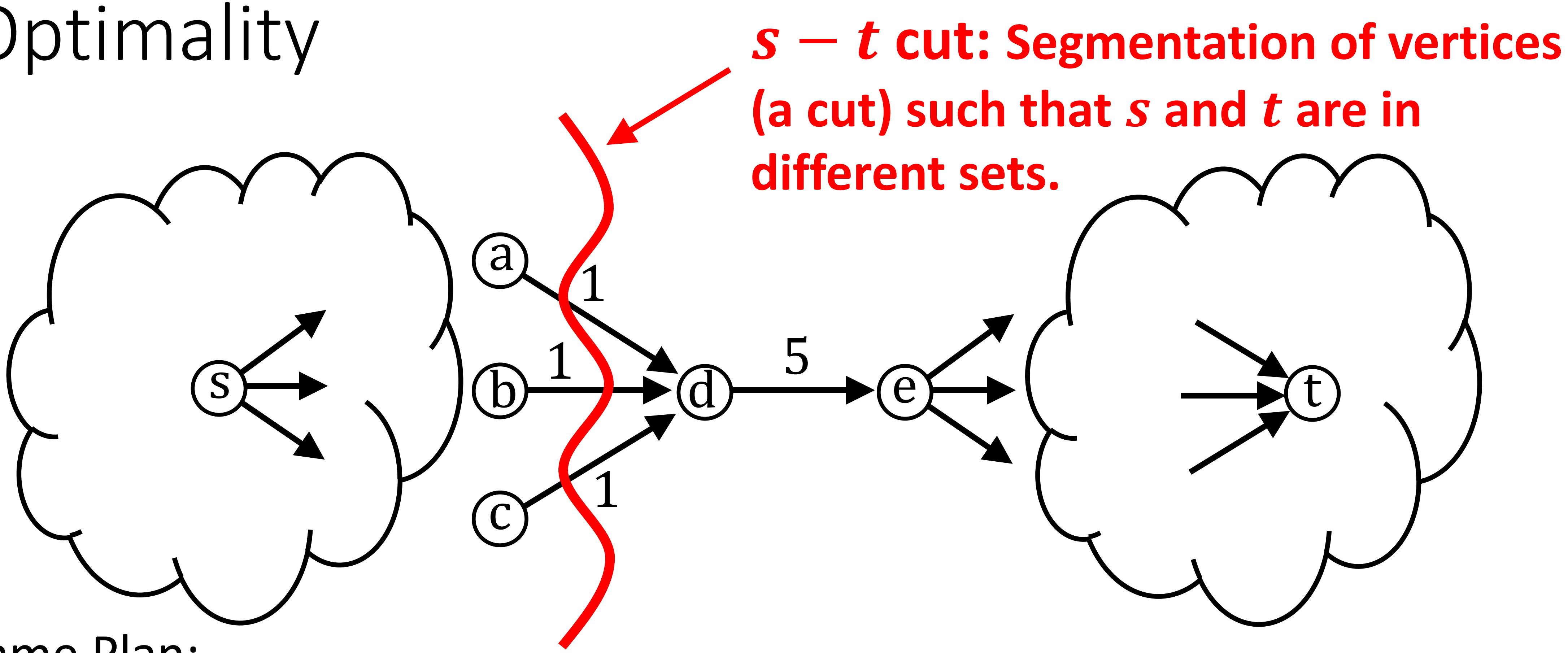

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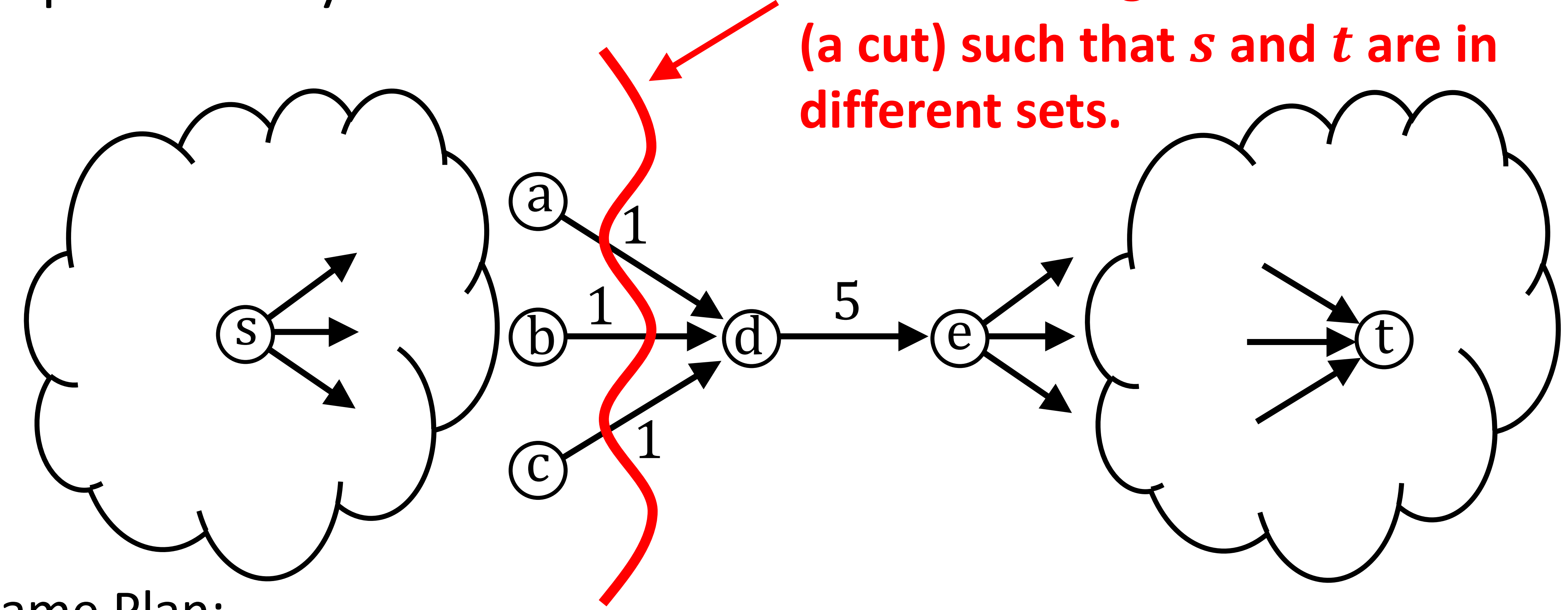

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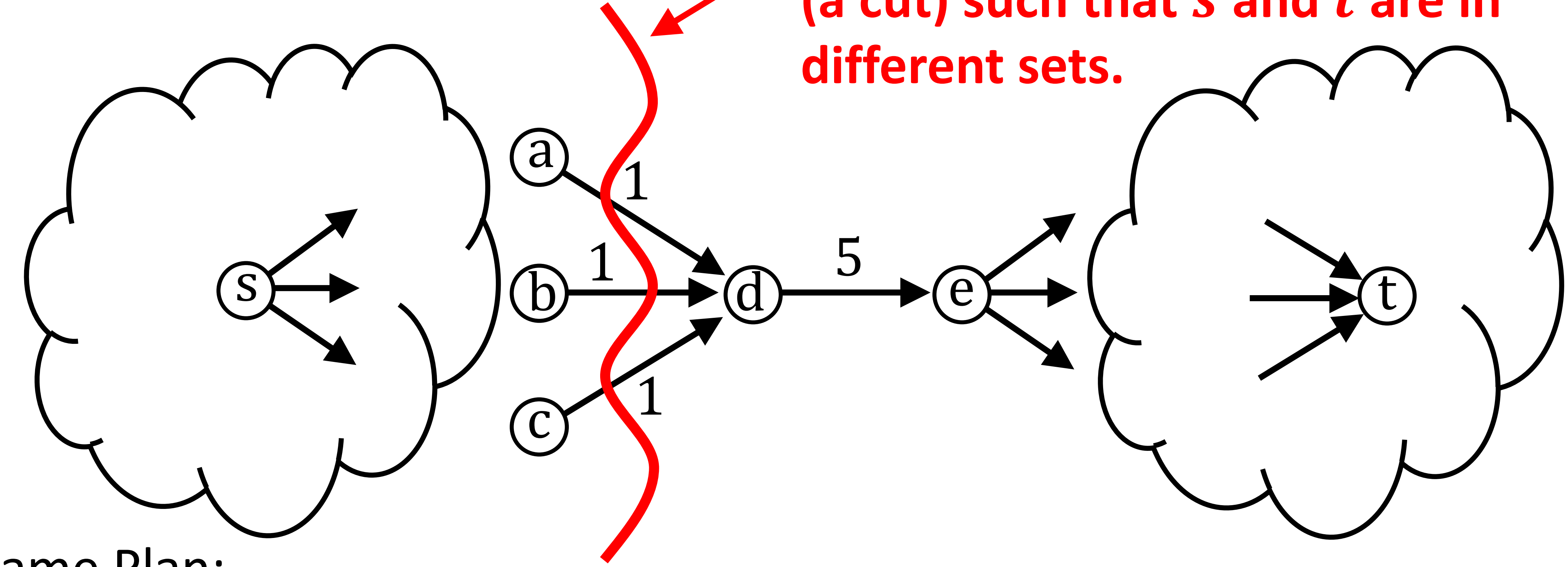

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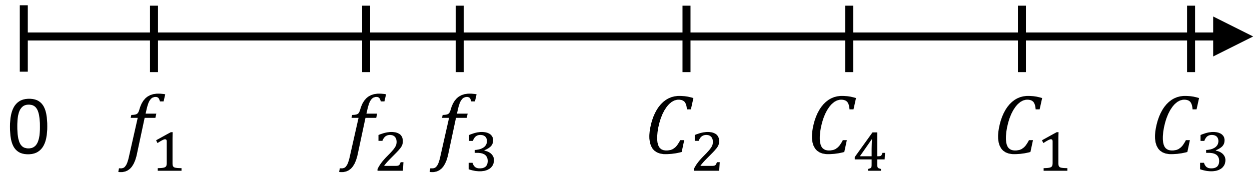

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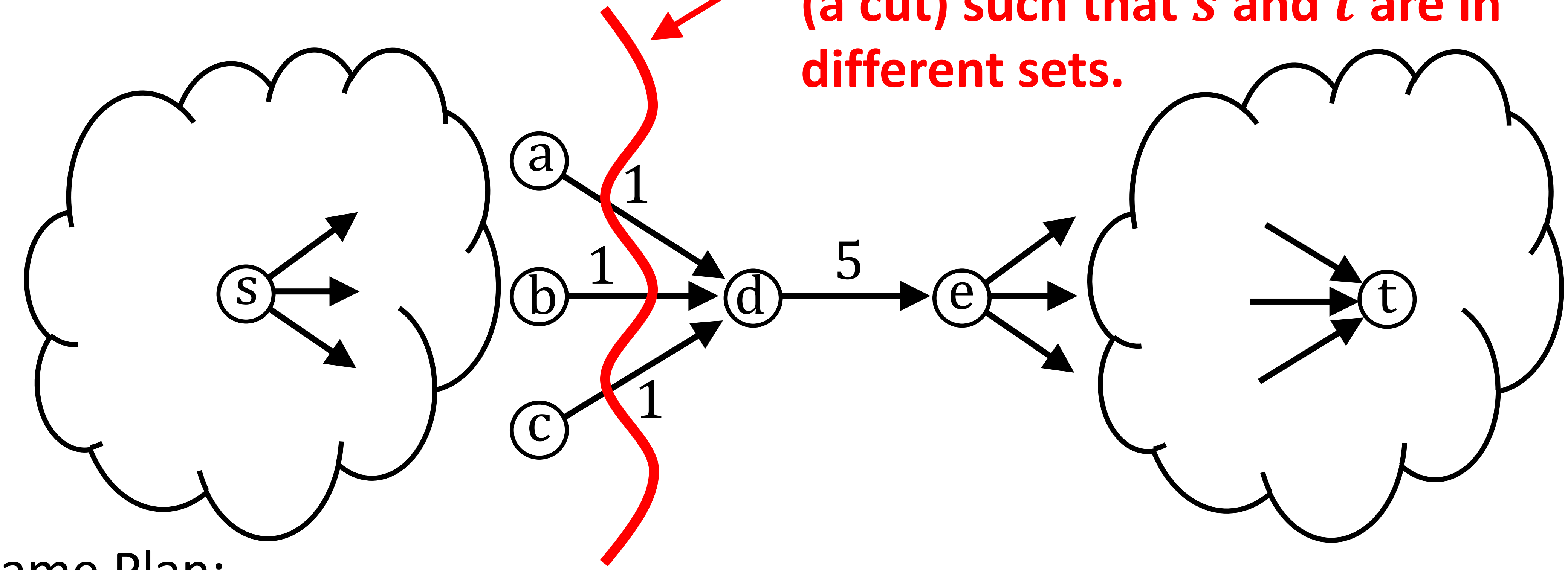

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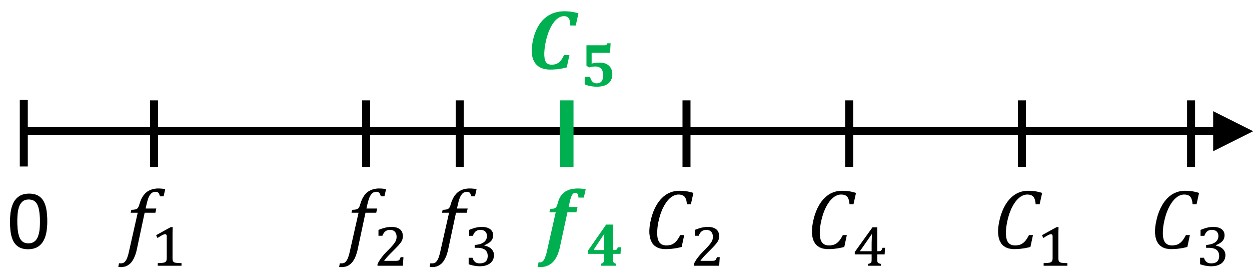

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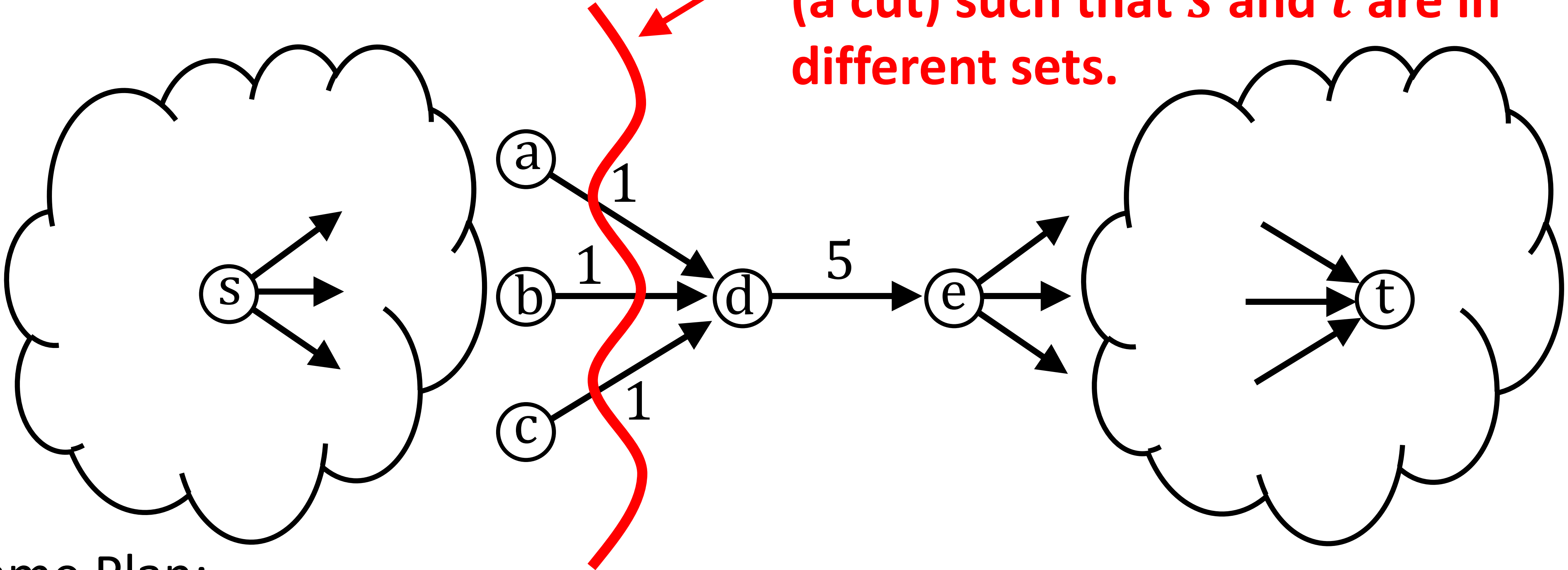

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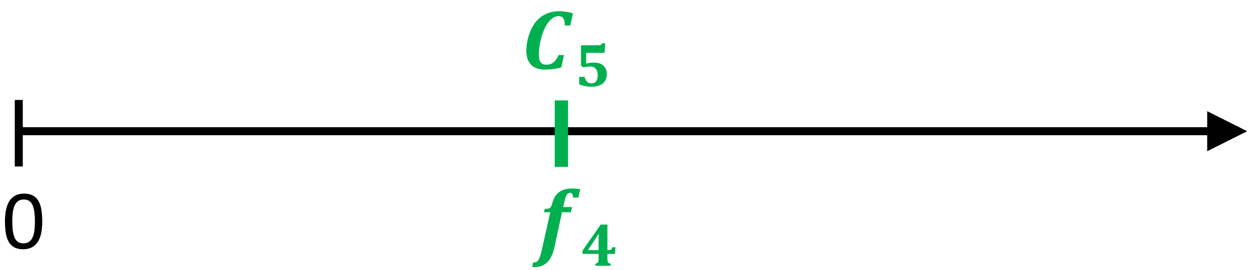

If we find some flow whose value equals the capacity of some cut, it must be the optimal flow.

Optimality

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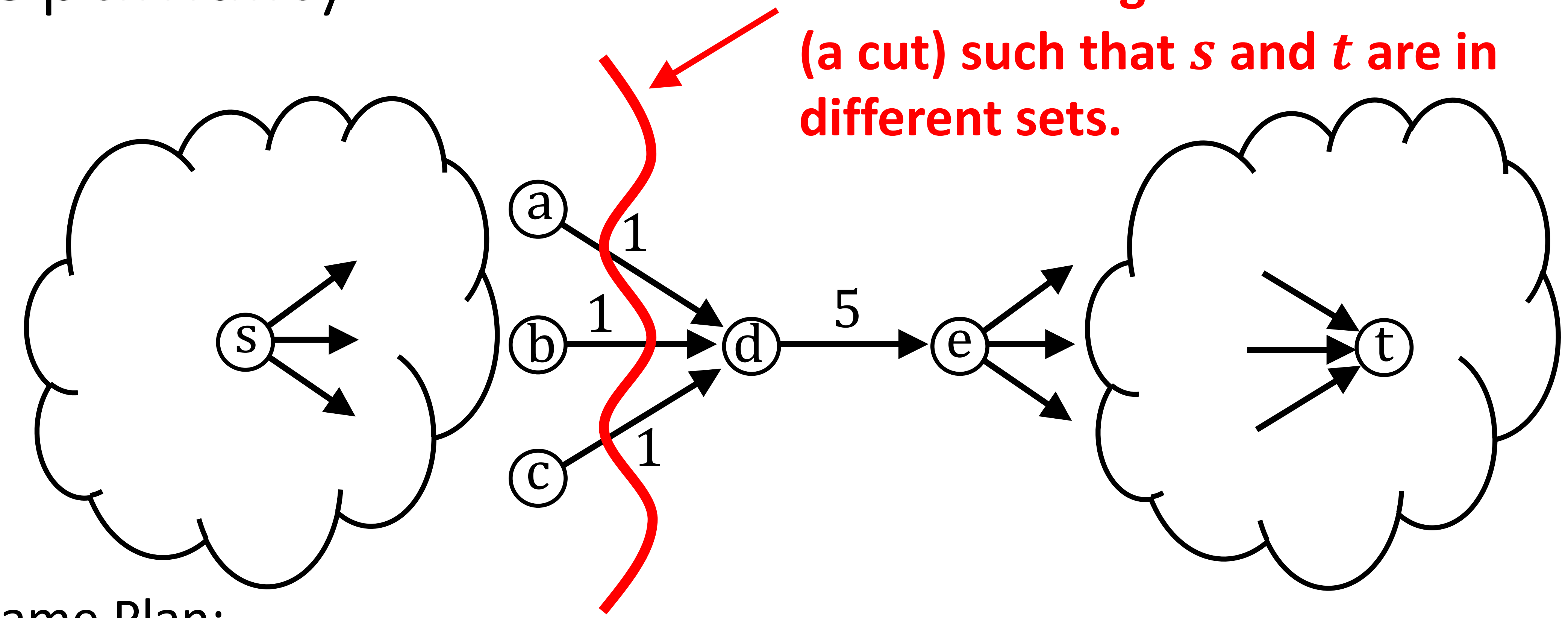

Game Plan:

1. Show that value of every flow is $\leq$ capacity of every cut.
2. Given a flow where there are no $s - t$ paths left in the residual graph, there is a specific cut whose capacity = flow value.

Optimality

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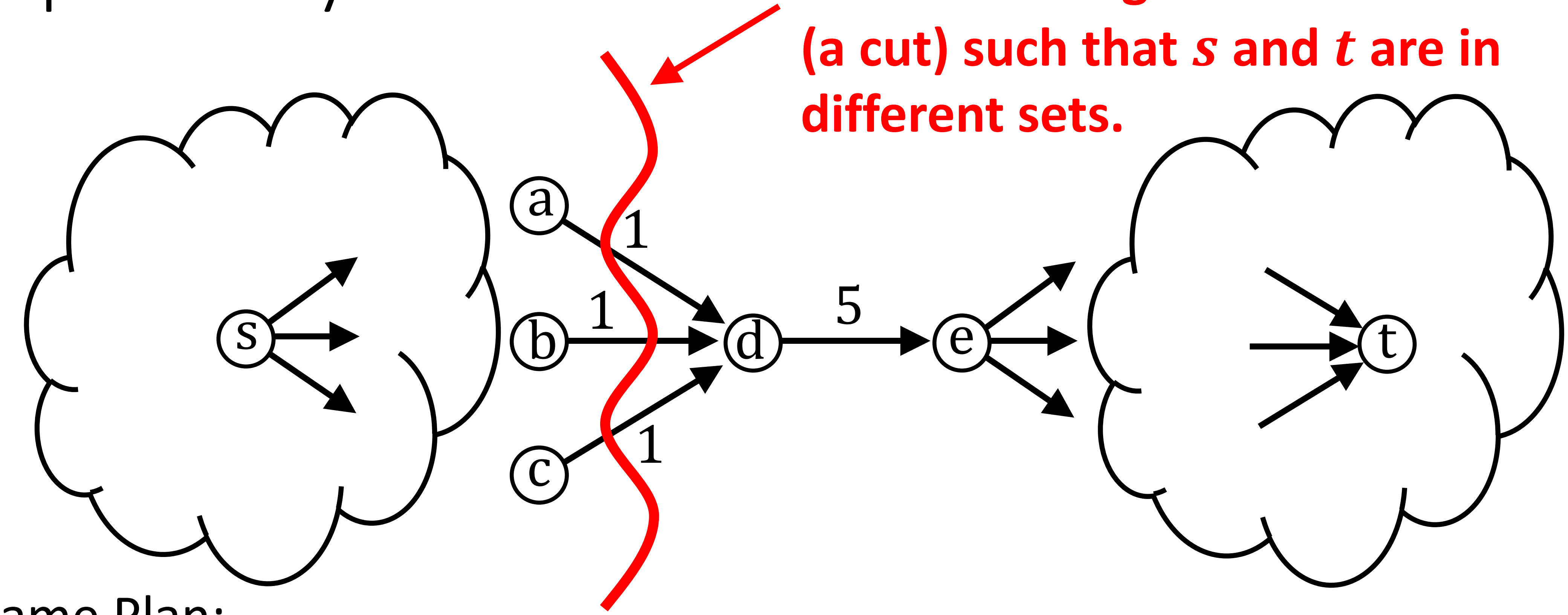

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$\Rightarrow$ **The algorithm is optimal**

Optimality

Theorem 1: Let G be a flow network, (A, B) be an $s - t$ cut, and f be an $s - t$ flow. Then, $|f| = \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{in}(A)} f(e)$.

Proof:

Edges that enter the set A

Edges that leave the set A

Optimality

This relates arbitrary $s-t$ flows to arbitrary $s-t$ cuts

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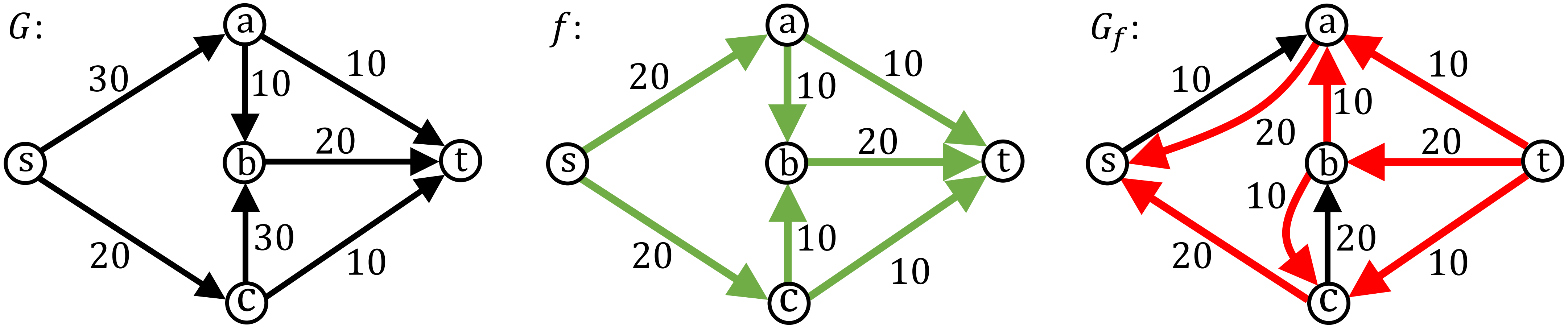
$|f| = 40$

<u>Proof:</u>



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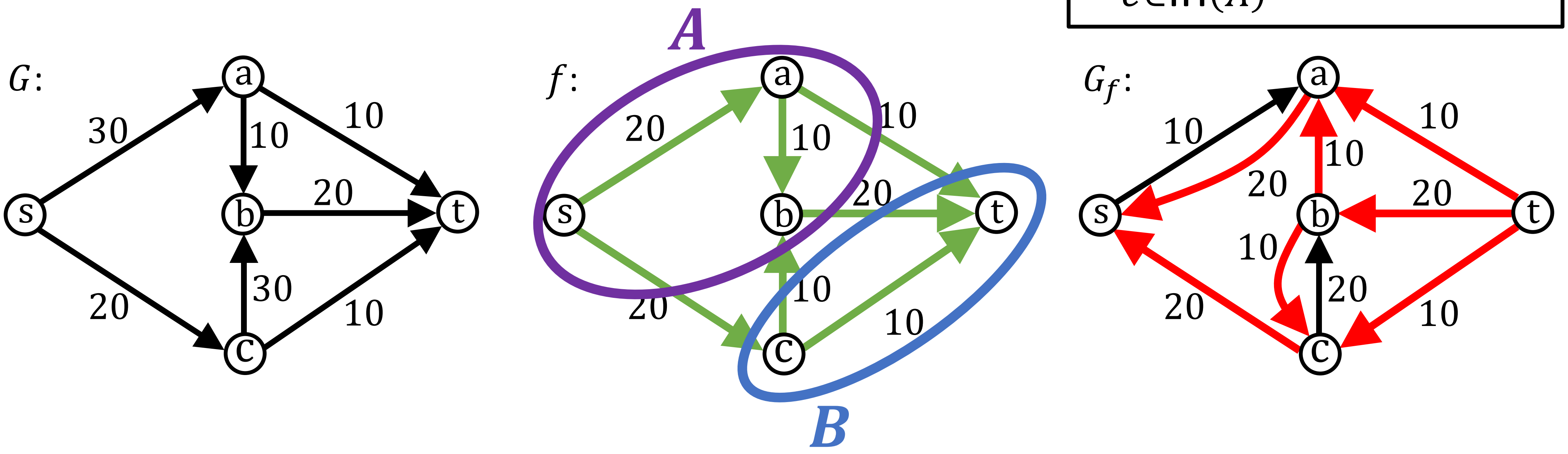
Proof:

$|f| = 40$
$\sum_{e \in \text{out}(A)} f(e) = 50$
$\sum_{e \in \text{in}(A)} f(e) = 10$



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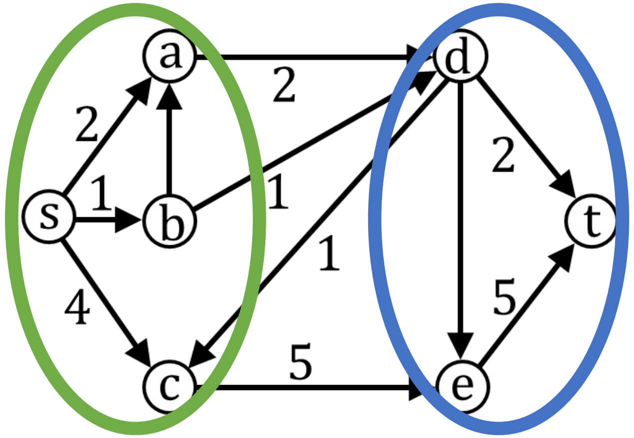

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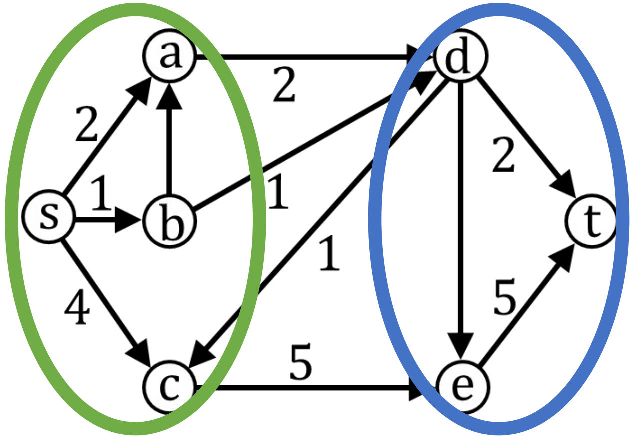
Then, $\forall v \in A, v \neq s, f^{out}(v) - f^{in}(v) = ?$



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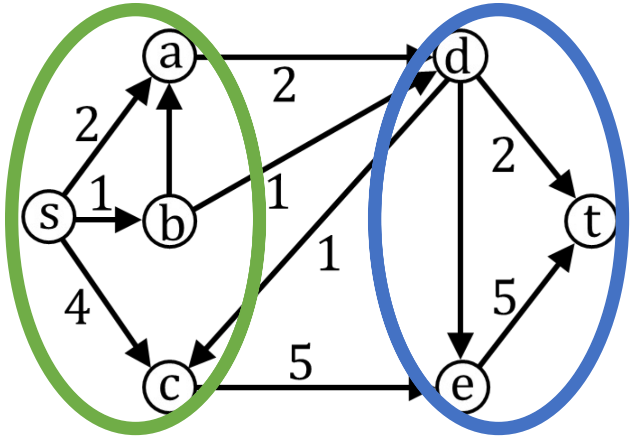
Then, $\forall v \in A, v \neq s$, $f^{out}(v) - f^{in}(v) = 0$ (by conservation of flow).



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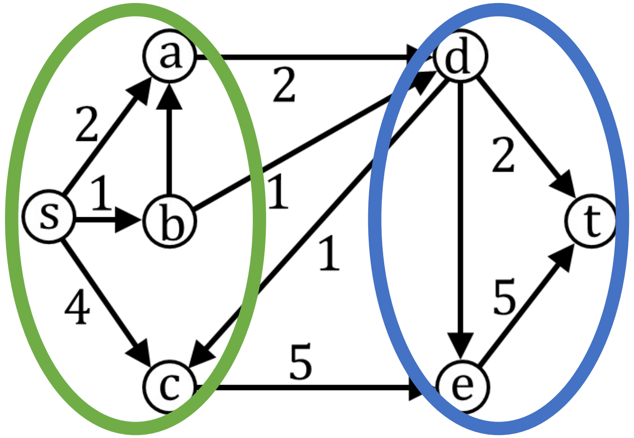

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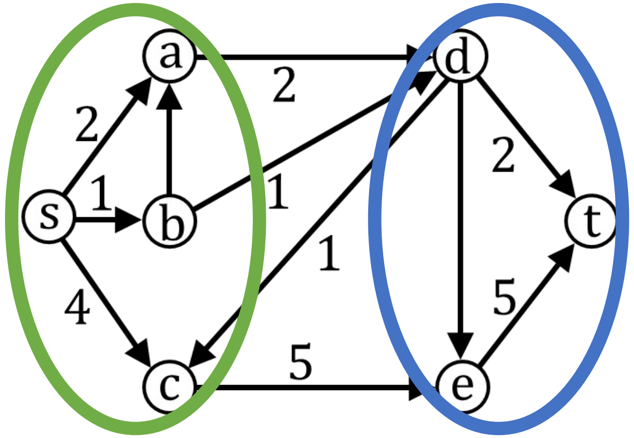
$\Rightarrow |f| = f^{out}(s) - f^{in}(s)$ (since $f^{in}(s) = 0$)



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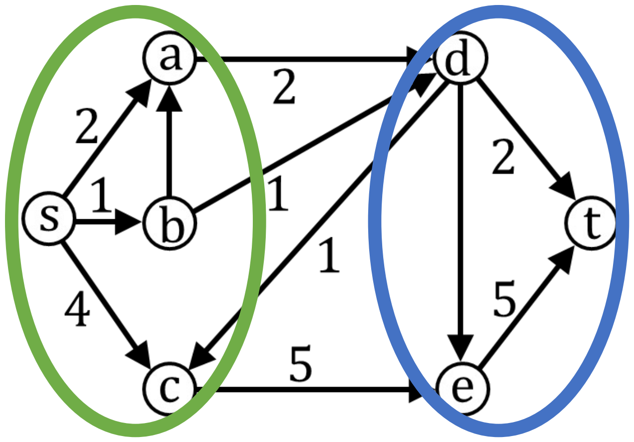
$\implies |f| = f^{out}(s) - f^{in}(s)$ (since $f^{in}(s) = 0$)

$\implies |f| = \sum_{v \in A}(f^{out}(v) - f^{in}(v))$ (Only $\neq 0$ when $v = s$).



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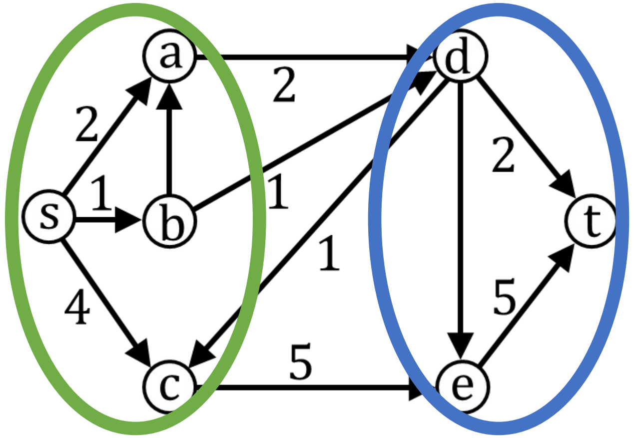

Optimality

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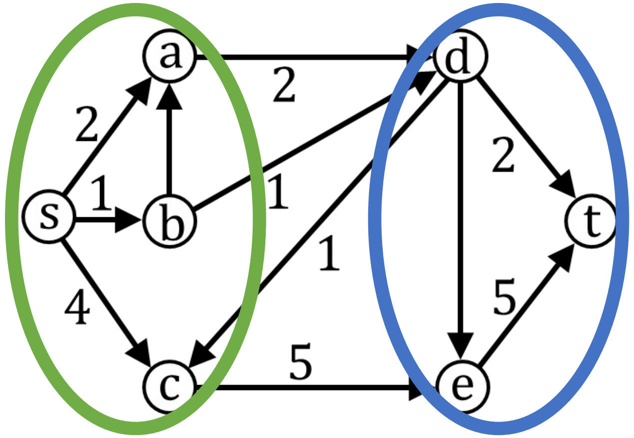

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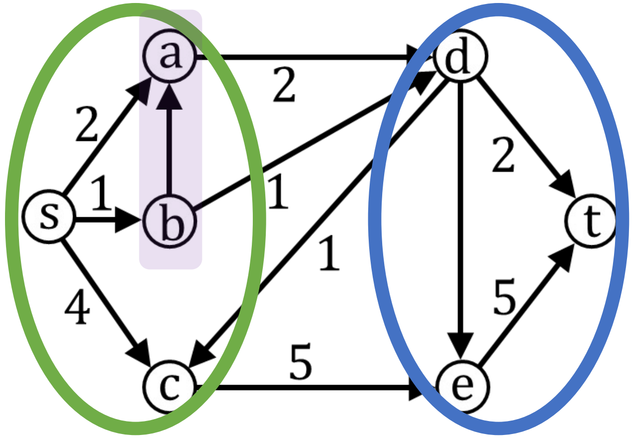

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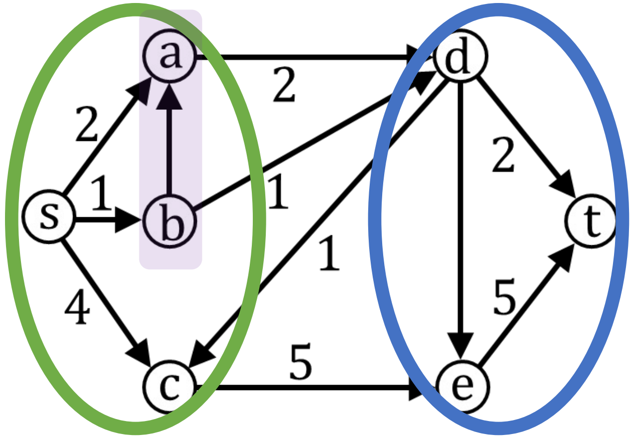

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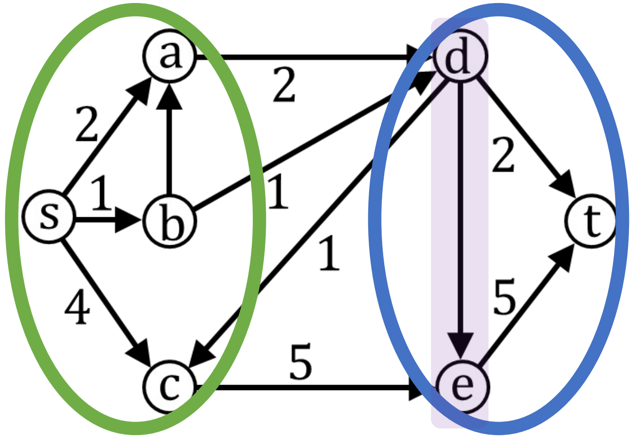

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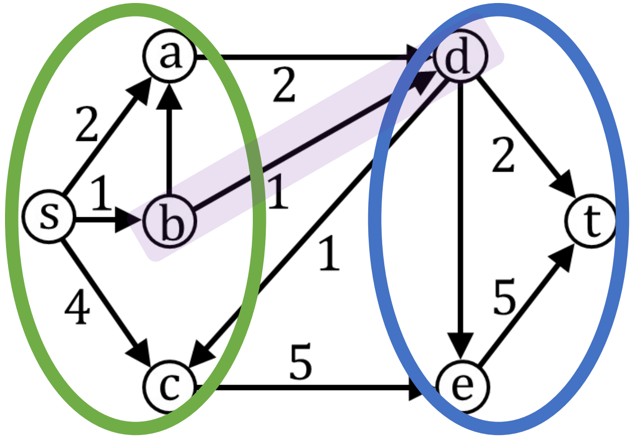

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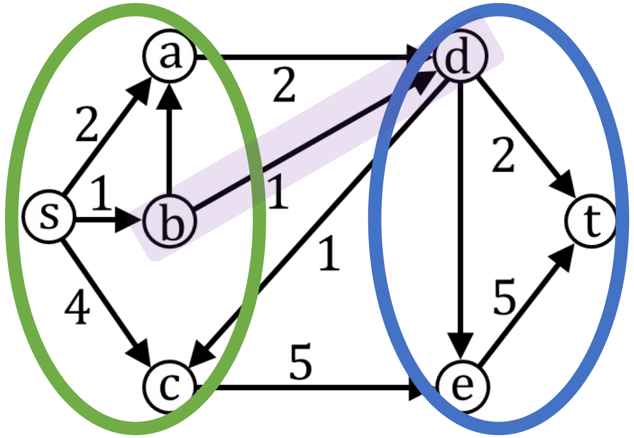

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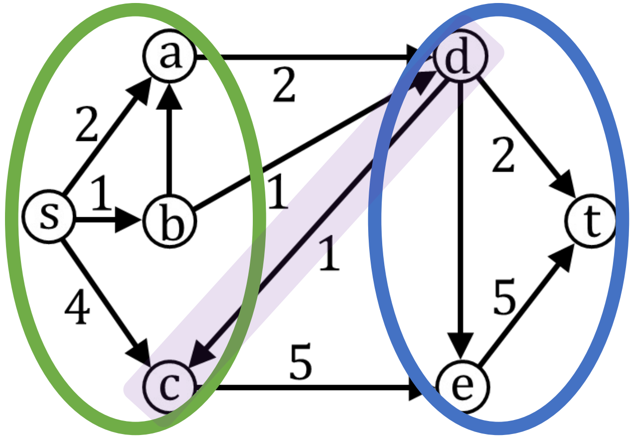

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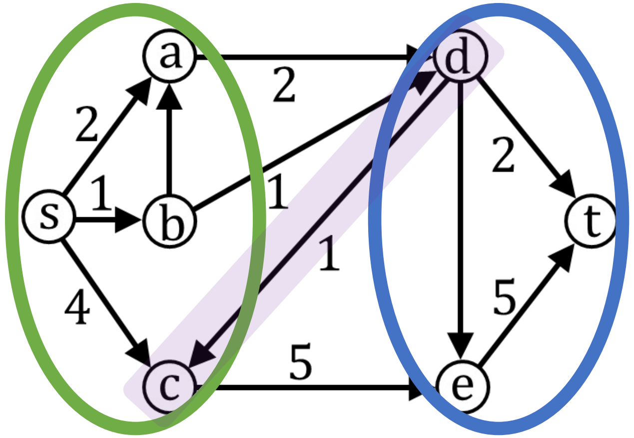

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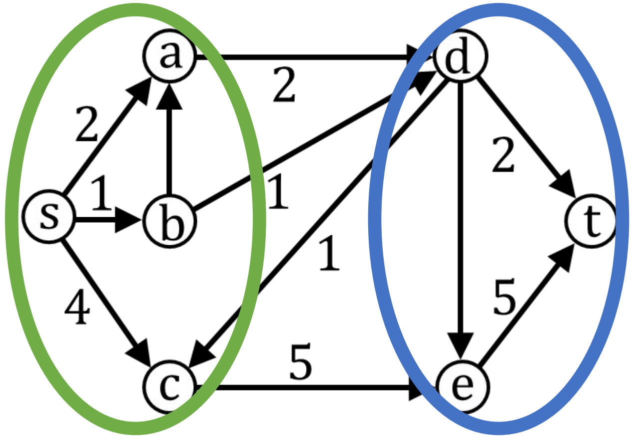

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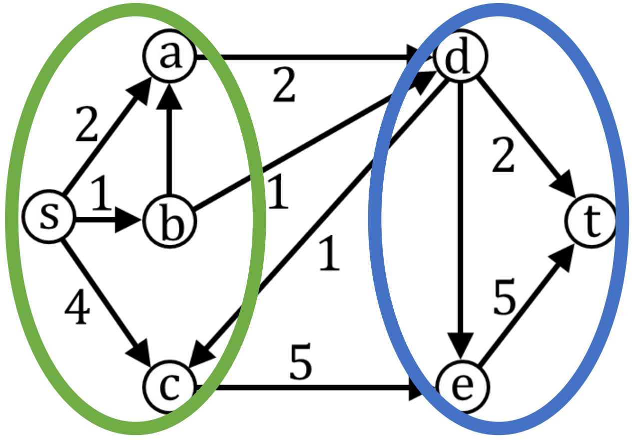

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Proof:

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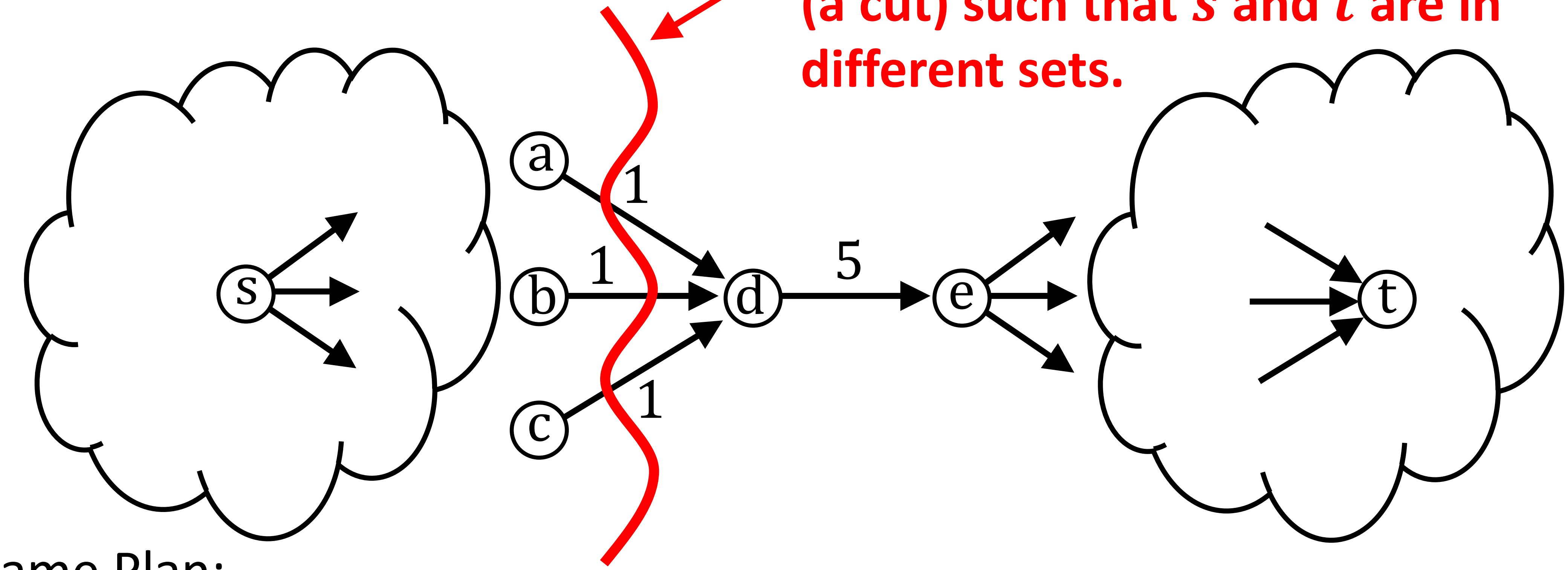
If we find some flow f and some cut (A, B) such that $|f| = c(A, B)$, then f is a maximum flow.

Optimality

$s - t$ cut: Segmentation of vertices (a cut) such that s and t are in different sets.



Game Plan:

1. Show that value of every flow is ≤ capacity of every cut.
2. Given a flow where there are no $s - t$ paths left in the residual graph, there is a specific cut whose capacity = flow value.

⇒ **The algorithm is optimal**

Optimality

Theorem 2: if f is an $s-t$ flow such that no $s-t$ path exists in residual graph G_f, then there is an $s-t$ cut (A, B) in $G=(V, E)$ for which $|f|=c(A, B)$.

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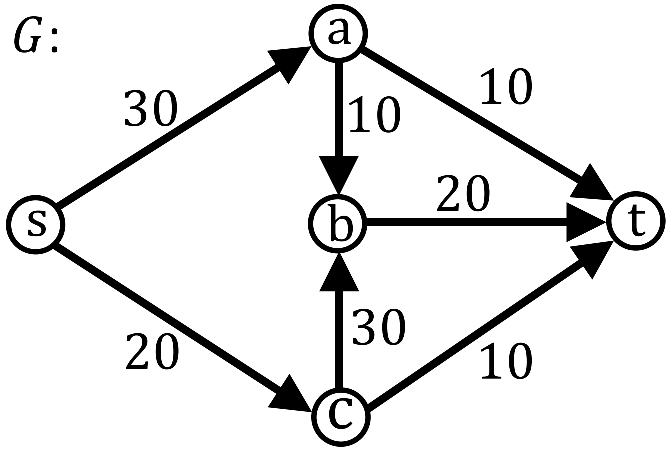

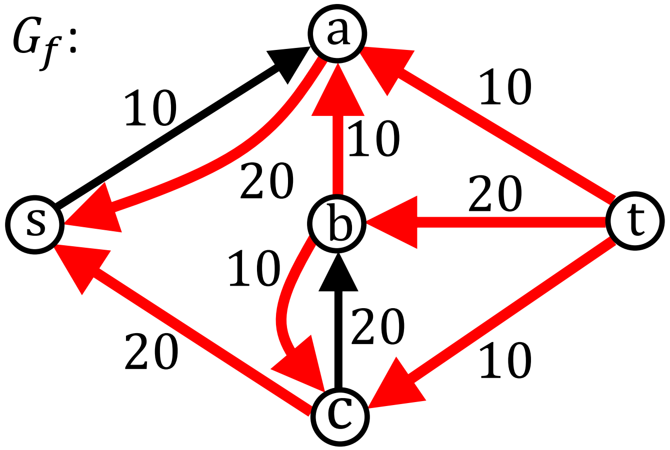

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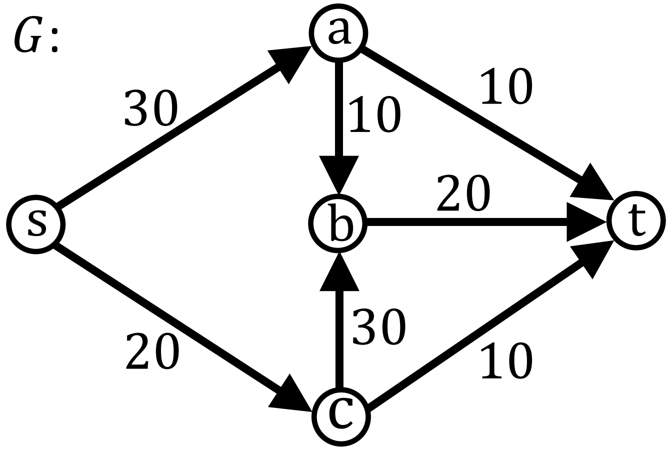

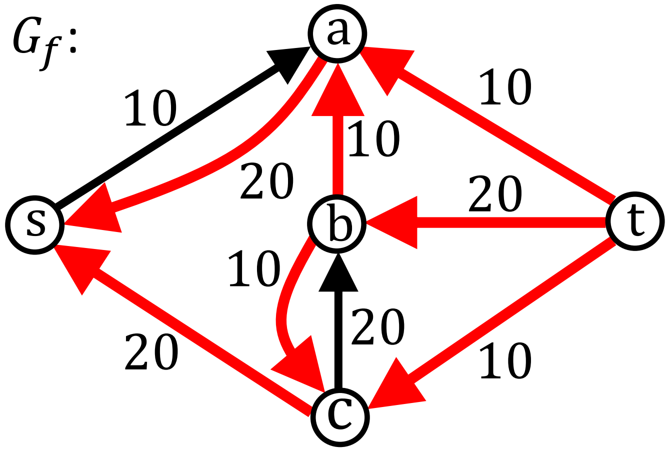

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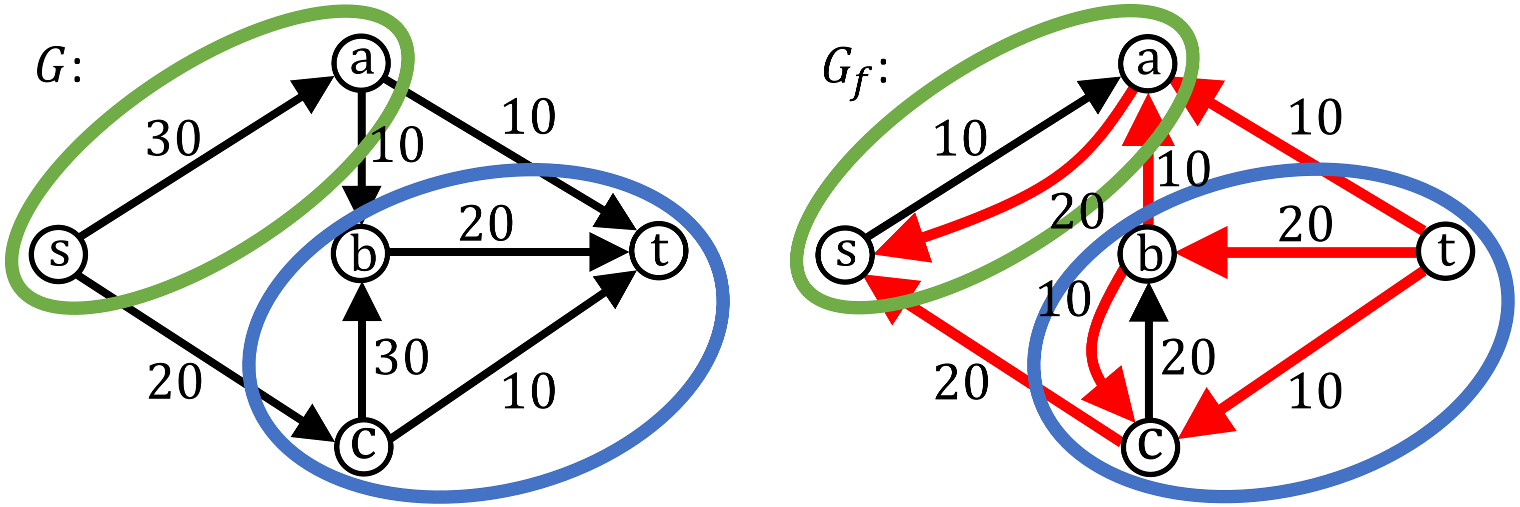

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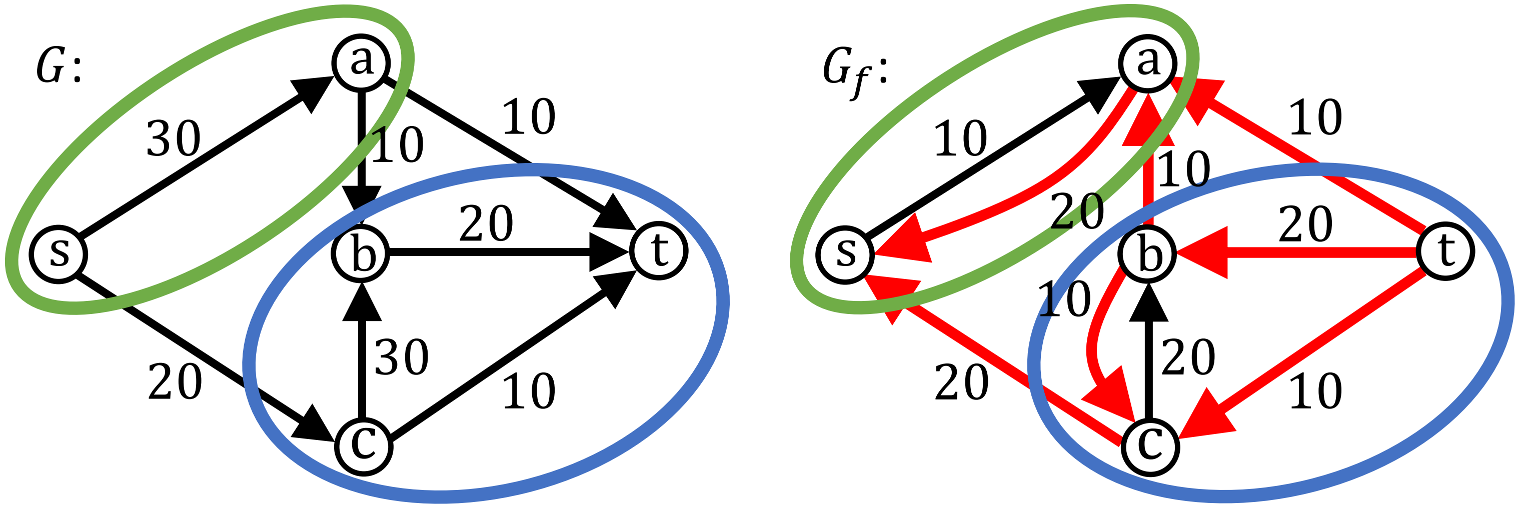

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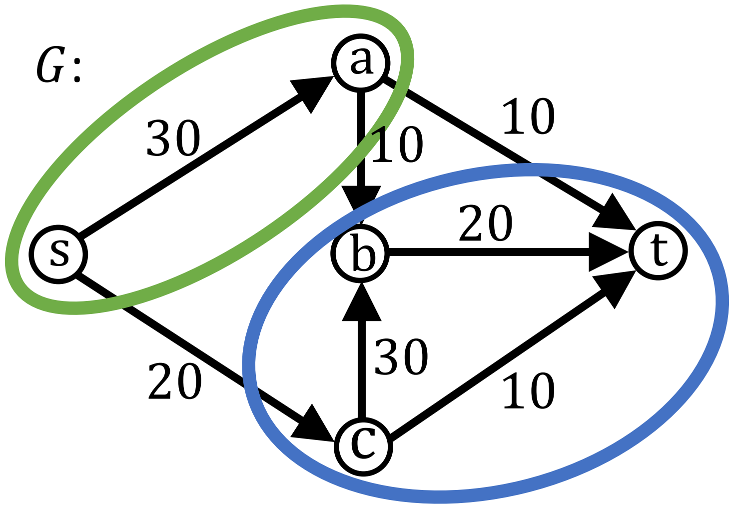
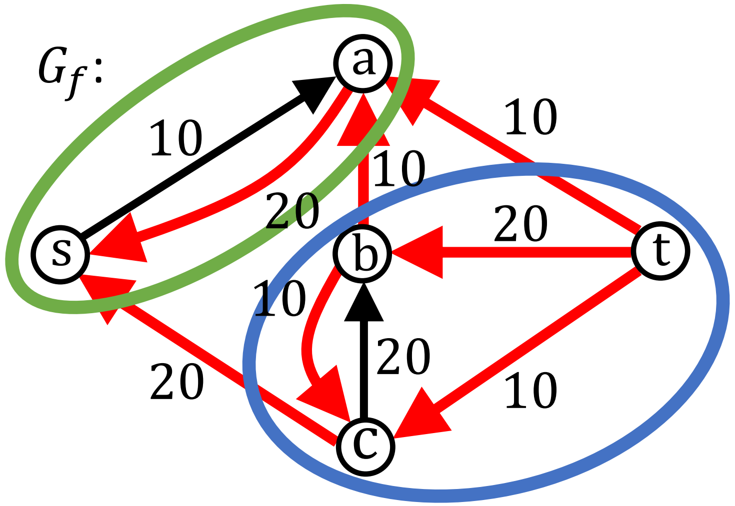
Need to compare flow across cut to capacity of cut.





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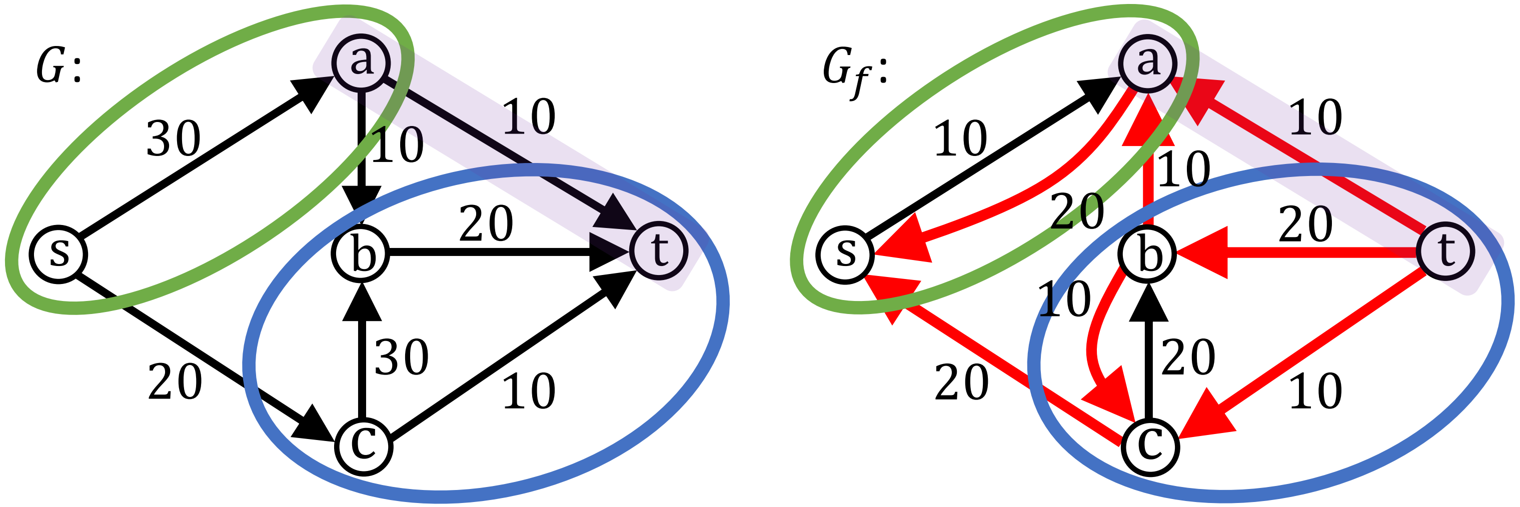
What can we say about $f(e)$ related to its capacity?



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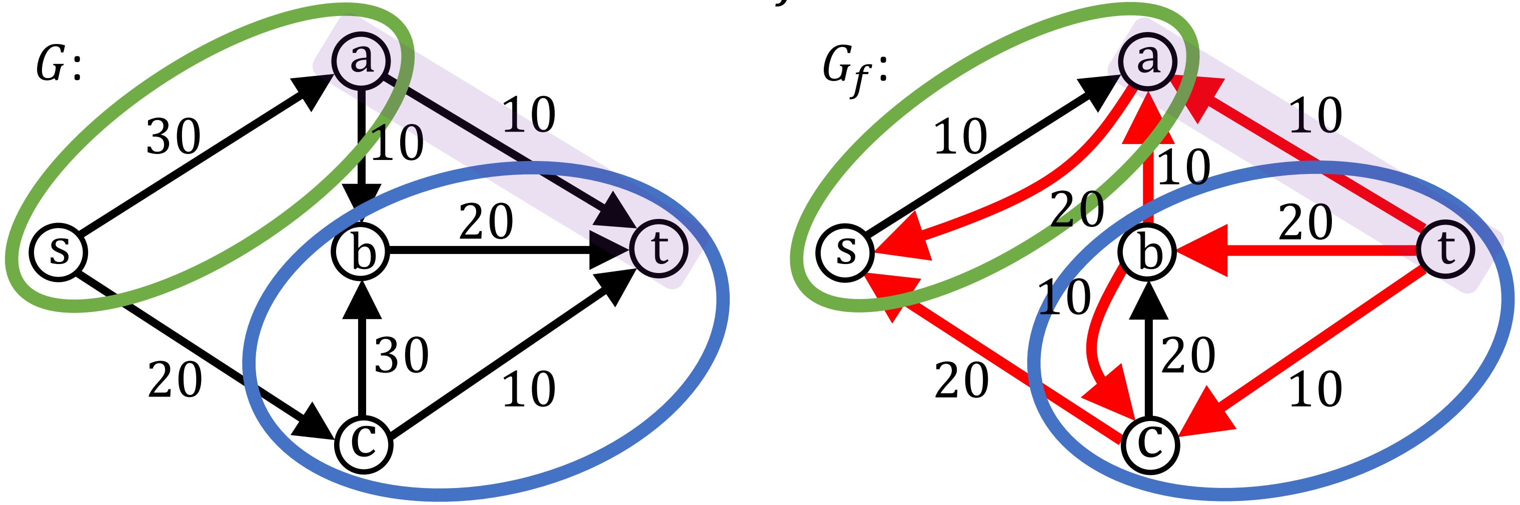

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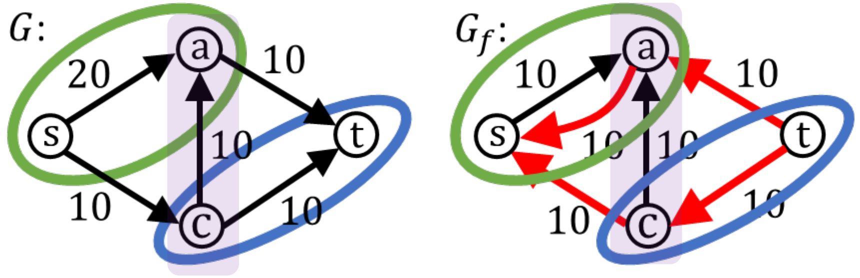
Let $e' = (u', v') \in E$ (directed edge) such that $u' \in B$ and $v' \in A$.

What can we say about $f(e')$?



Optimality

<u>Theorem 2:</u> if f is an $s-t$ flow such that no $s-t$ path exists in residual graph G_f, then there is an $s-t$ cut (A, B) in $G = (V, E)$ for which $|f| = c(A, B)$.

<u>Proof:</u> Let $A = \{v \in V: \exists\ s - v$ path in $G_f\}$ and $B = V \backslash A$.

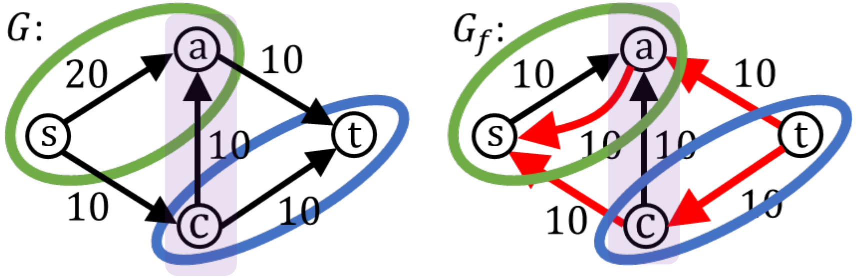
(A, B) is an $s-t$ cut (because it partitions V, $s \in A$, and $t \in B$)

Let $e = (u, v) \in E$ (directed edge) such that $u \in A$ and $v \in B$.

$f(e) = c_e$ (since $(u, v) \notin G_f$, otherwise v would be in A)

Let $e' = (u', v') \in E$ (directed edge) such that $u' \in B$ and $v' \in A$.

$f(e') = 0$ (since $(v', u') \notin G_f$, otherwise u' would be in A)



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Therefore, $|f|=\sum_{e\in \text{out}(A)} f(e)-\sum_{e\in \text{in}(A)} f(e)$ (Theorem 1)

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$= \sum_{e \in \text{out}(A)} c_e - 0 = c(A, B)$

Optimality

Theorem: The flow returned by the Ford-Fulkerson algorithm is a maximum flow.

Proof:

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Corollary: Suppose G is a flow network, f is an $s-t$ flow on G, and (A,B) is an $s-t$ cut. Then, $|f| \leq c(A,B)$. (i.e. *every* flow is bounded by *any* $s-t$ cut)

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Proof: Since the Ford-Fulkerson algorithm finishes when no $s - t$ paths remain in G_f, Theorem 2 says there must be an $s - t$ cut such that the value of flow found equals the capacity of the cut.

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