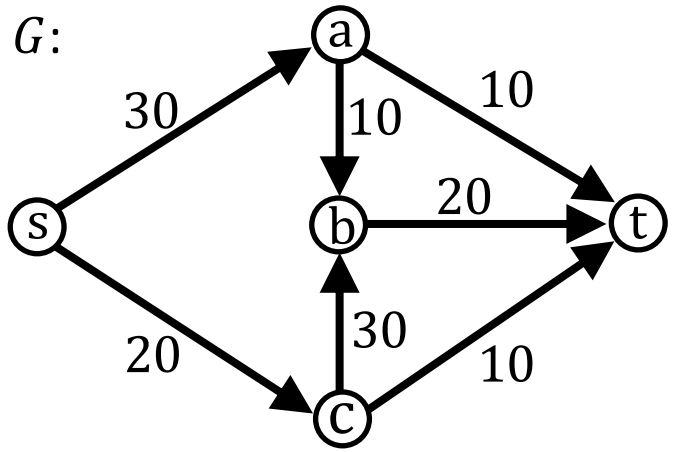
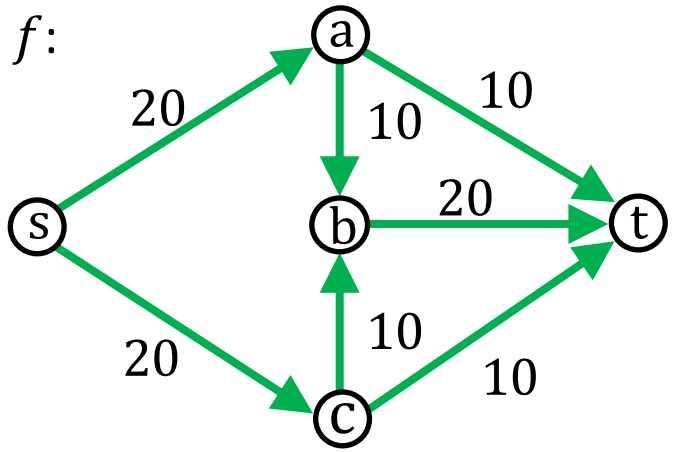
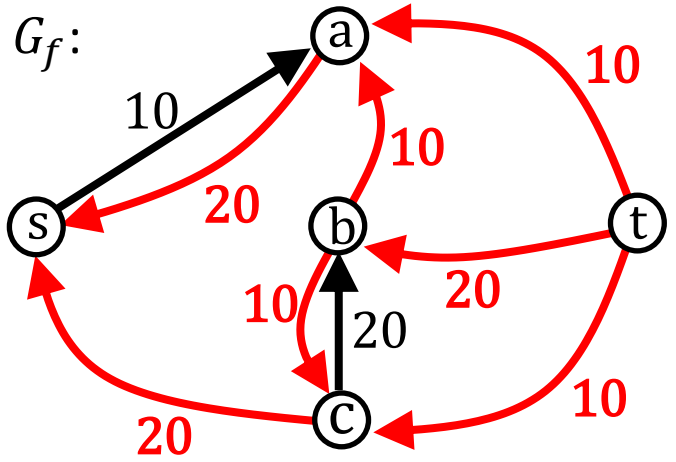
Flow Networks
CSCI 532

Ford-Fulkerson





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Max-Flow(G)
   f(e) = 0 for all e in G
   while s-t path in Gf exists
      P = simple s-t path in Gf
      f'= augment(f, P)
      f = f'
      Gf = Gf'
   return f
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augment(f, P)
   b = bottleneck(P,f)
   for each edge (u, v) in P
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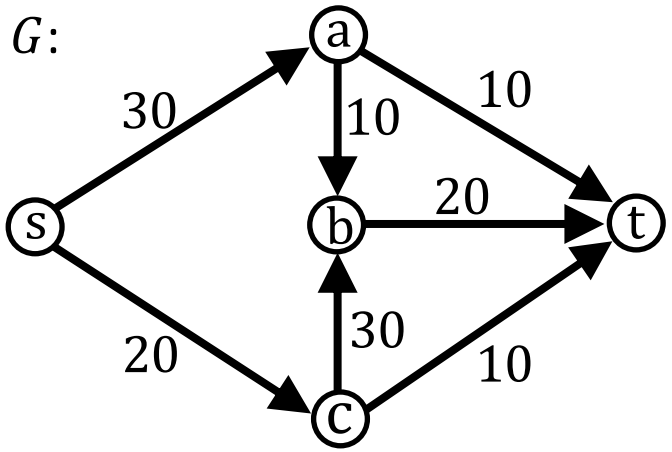

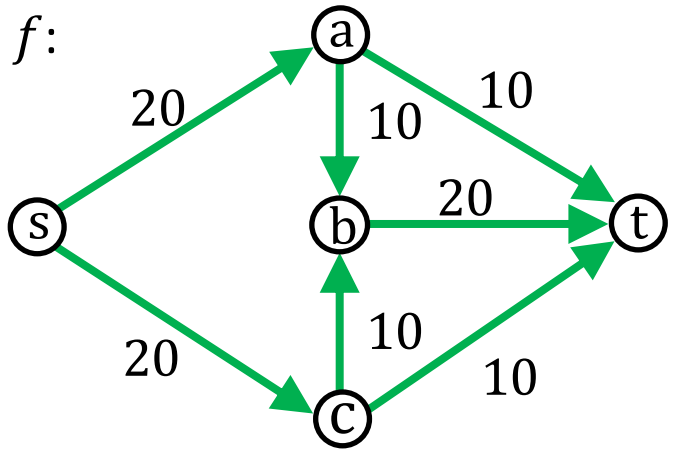

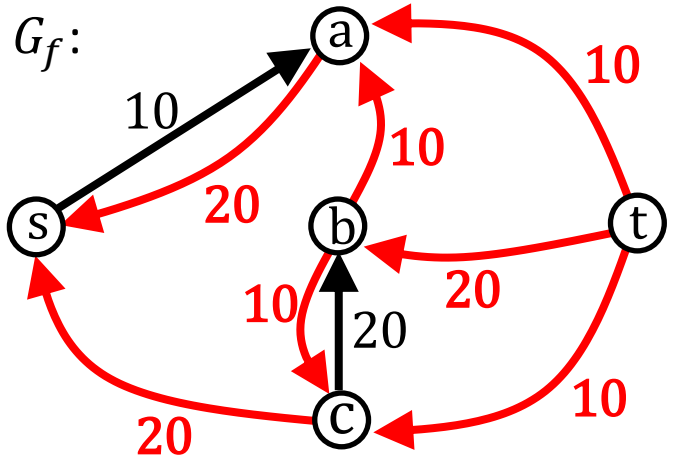


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Need to show:

1. Validity.
2. Running time.
3. Finds max flow.

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Claims:

1. If edge capacities are integer-valued, flows found by algorithm will be too.

2. If edge capacities are integer-valued, the algorithm will terminate.

3. If an iteration starts with a valid flow, it ends with a valid flow.

4. The first iteration starts with a valid flow.

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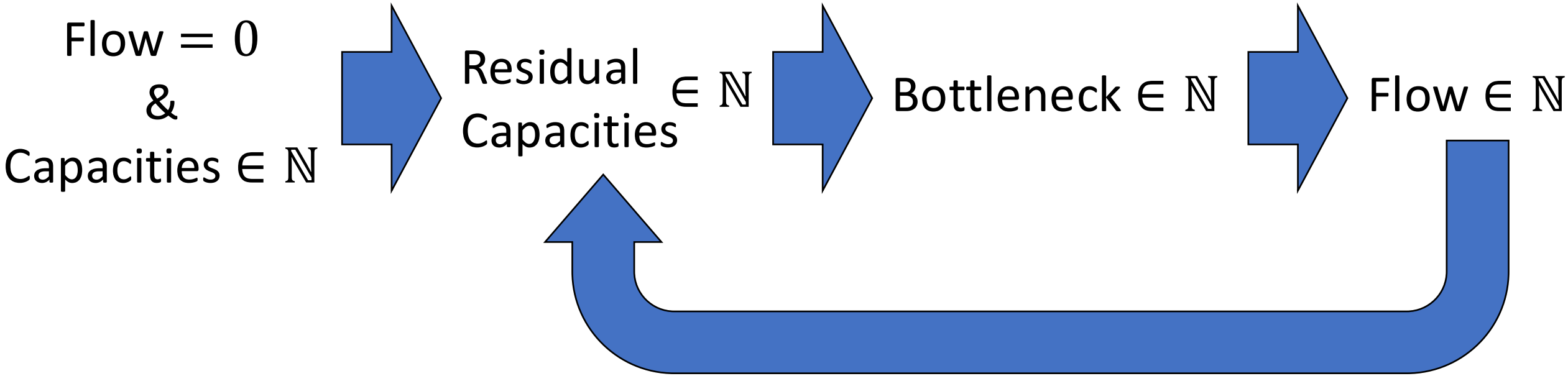

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   With integer capacities, each iteration increases flow by ≥ 1.

Bottleneck $\in \mathbb{N}$

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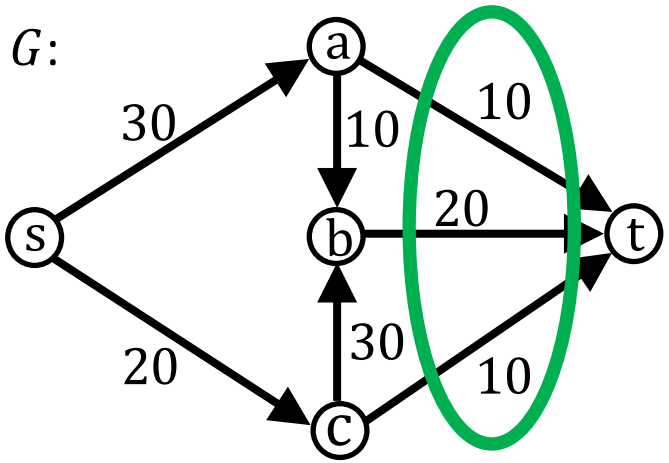

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Note: This does not hold for general edge capacities (i.e., irrational edge capacities *can* lead to non-terminating scenarios).

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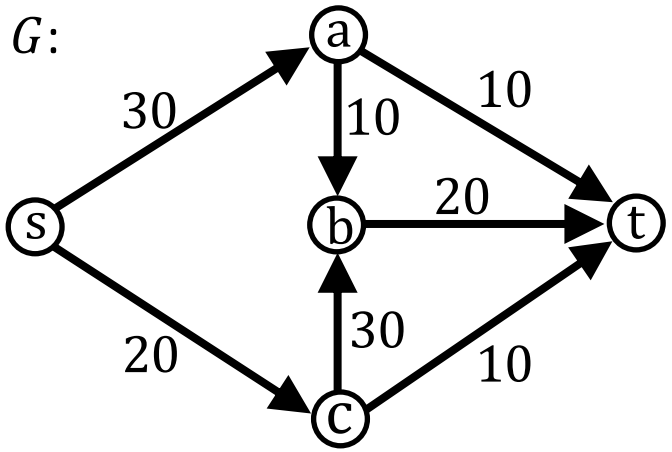

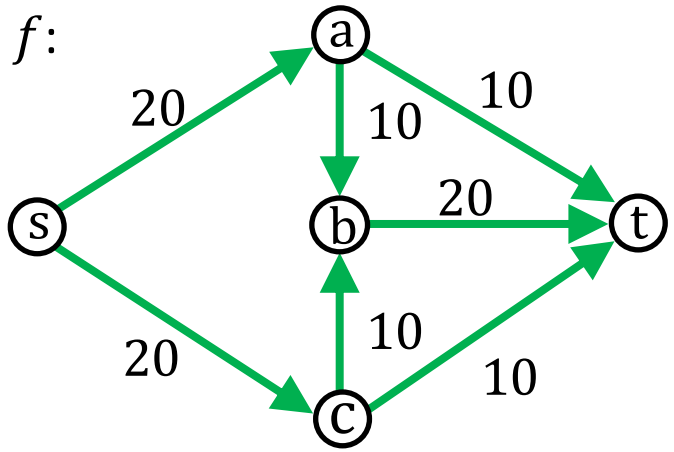

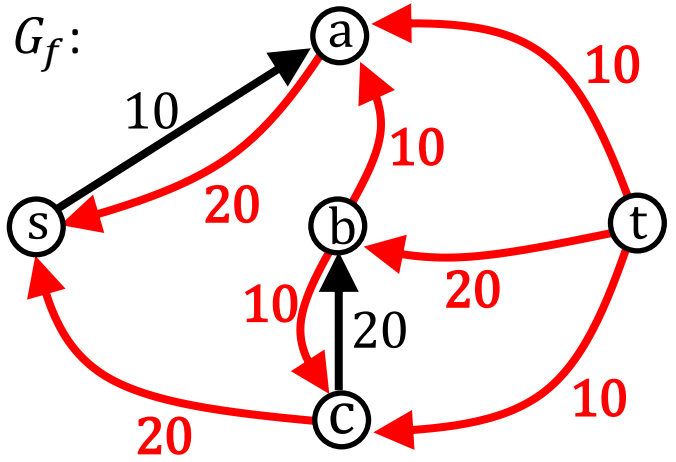


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We only need to consider nodes/edges on the path. Other nodes/edges aren't modified.

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Can't overflow capacities on back edges, because…

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Can't overflow capacities on back edges, because we are removing flow from them. So, if they were valid, less flow doesn't change that.

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Can't overflow capacities on forward edges since…

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Can't overflow capacities on forward edges since bottleneck = $\min_e(c_e - f_e)$

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$\leq f_e + c_e - f_e = c_e$

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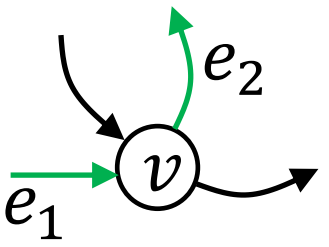
Considering a node on the path. Suppose the edge in is a forward edge and the edge out is a forward edge.



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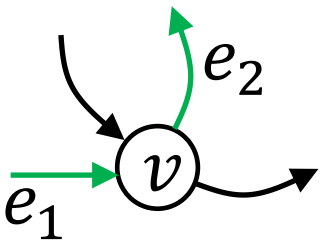
Then, $f'_{in} =$



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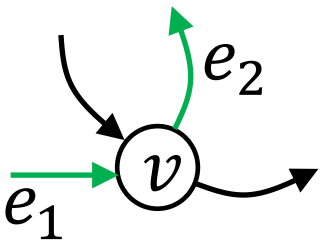

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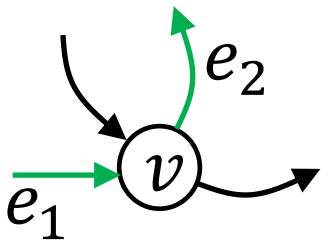

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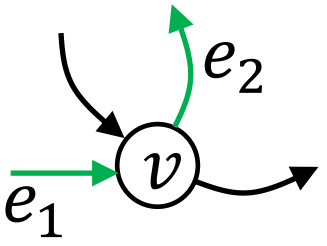

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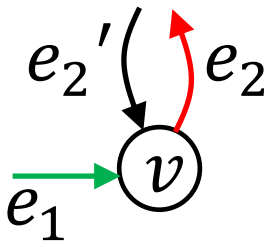

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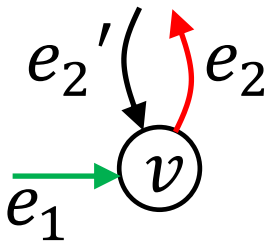

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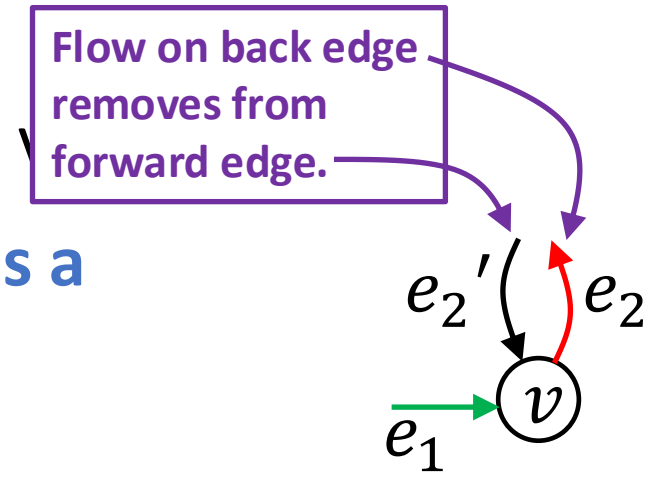

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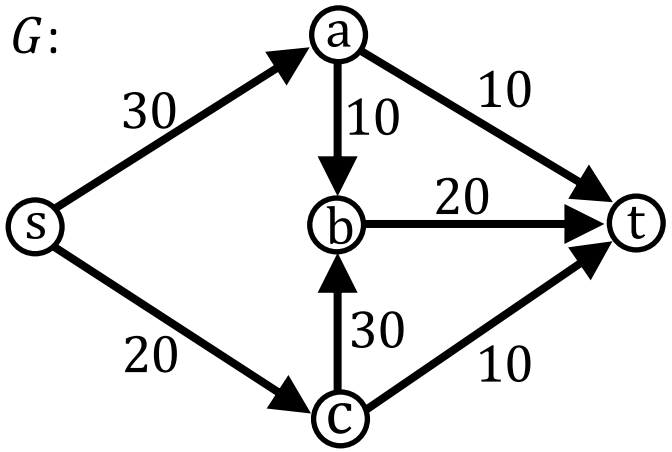

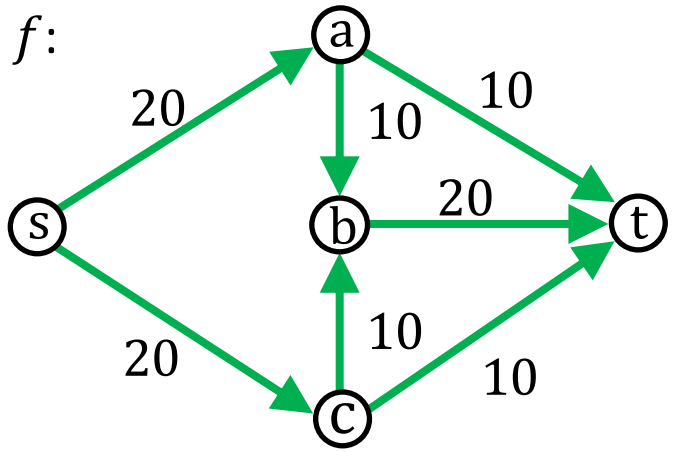

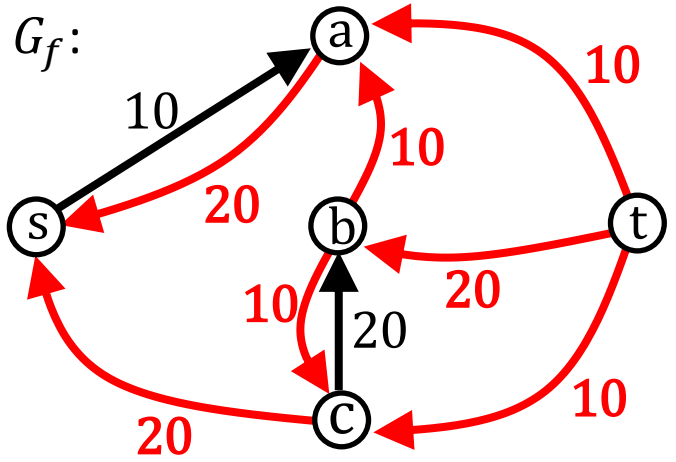


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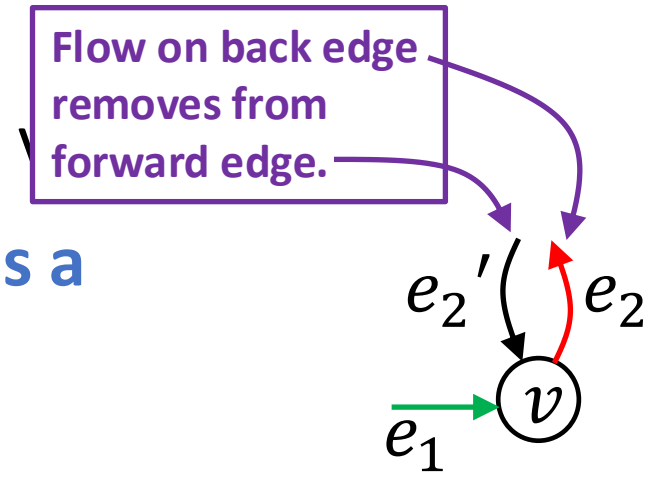

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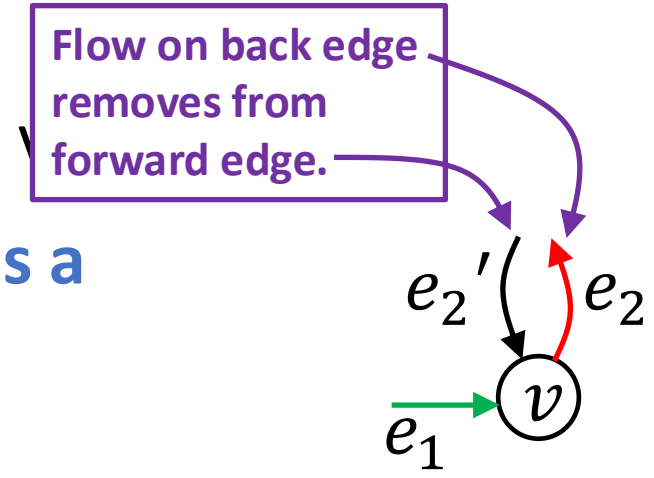

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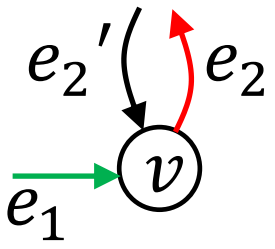

Then, $f'_{in} = f_{in}$ + bottleneck – bottleneck $= f_{in} = f_{out} = f'_{out}$

Ford-Fulkerson

Claims:

1. If edge capacities are integer-valued, flows found by algorithm will be too.
   Bottleneck values will always be integers, so resulting flows are too.
2. If edge capacities are integer-valued, the algorithm will terminate.
   With integer capacities, each iteration increases flow by ≥ 1. Since the max flow is bounded (e.g., by total capacity into t), the algorithm needs $\leq$ |Max Flow| iterations.
3. If an iteration starts with a valid flow, it ends with a valid flow.

Considering a node on the path. Suppose the edge in is a back edge and the edge out is a forward edge.

Considering a node on the path. Suppose the edge in is a back edge and the edge out is a back edge.

Ford-Fulkerson

Claims:

1. If edge capacities are integer-valued, flows found by algorithm will be too.
   Bottleneck values will always be integers, so resulting flows are too.
2. If edge capacities are integer-valued, the algorithm will terminate.
   With integer capacities, each iteration increases flow by ≥ 1. Since the max flow is bounded (e.g., by total capacity into t), the algorithm needs $\leq$ |Max Flow| iterations.
3. If an iteration starts with a valid flow, it ends with a valid flow.
   Each iteration sends flow along a residual $s - t$ path without violating capacities or conservation of flow. So, the resulting flow is valid.

Ford-Fulkerson

Claims:

1. If edge capacities are integer-valued, flows found by algorithm will be too.
   Bottleneck values will always be integers, so resulting flows are too.
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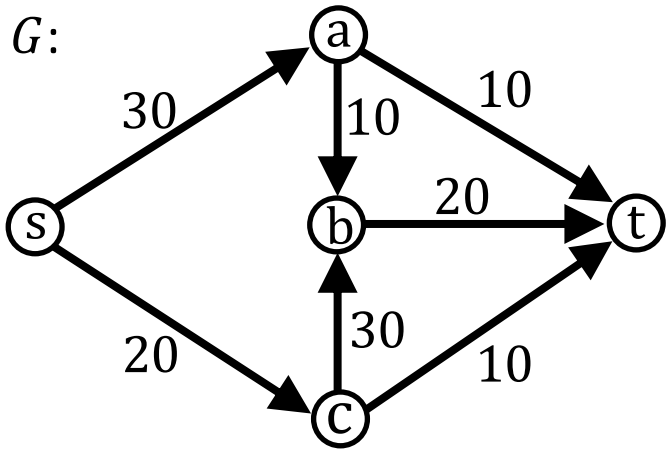
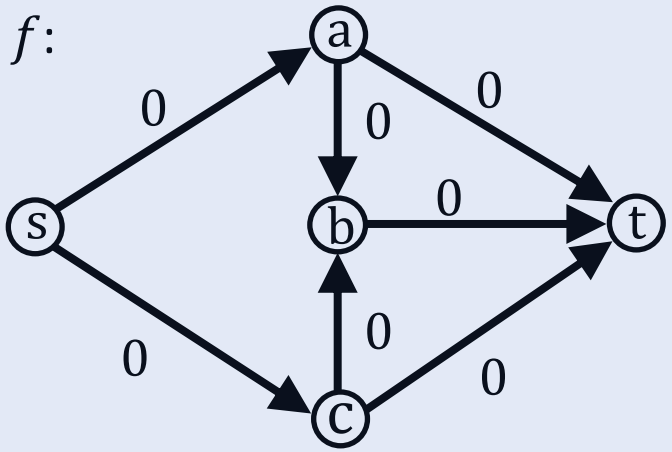
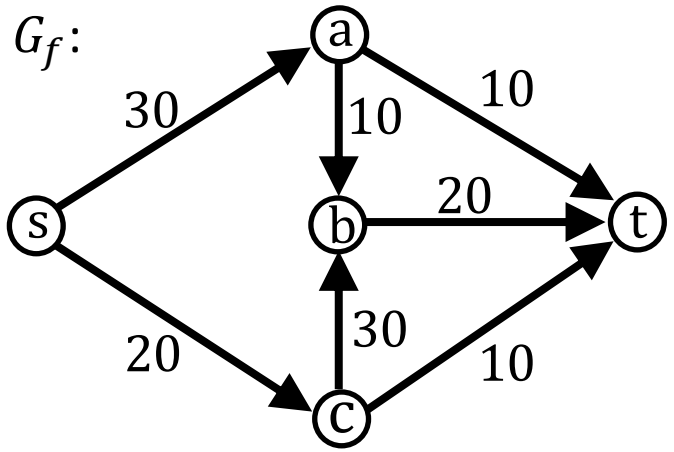
3. If an iteration starts with a valid flow, it ends with a valid flow.
   Each iteration sends flow along a residual $s - t$ path without violating capacities or conservation of flow. So, the resulting flow is valid.
4. The first iteration starts with a valid flow.

Max Flow Algorithm







Algorithm Overview

1. Start with 0 flow and initial residual graph.
2. Select an $s - t$ path P in residual graph.
3. Push bottleneck(P) flow on P.
4. Update residual graph.
5. Repeat until no $s - t$ paths exist in residual graph.

Ford-Fulkerson

Claims:

1. If edge capacities are integer-valued, flows found by algorithm will be too.
   Bottleneck values will always be integers, so resulting flows are too.
2. If edge capacities are integer-valued, the algorithm will terminate.
   With integer capacities, each iteration increases flow by ≥ 1. Since the max flow is bounded (e.g., by total capacity into t), the algorithm needs $\leq$ |Max Flow| iterations.
3. If an iteration starts with a valid flow, it ends with a valid flow.
   Each iteration sends flow along a residual $s - t$ path without violating capacities or conservation of flow. So, the resulting flow is valid.
4. The first iteration starts with a valid flow.
   0 flow on all edges meets capacity and conservation of flow constraints.

Ford-Fulkerson

What can be concluded from all this?

Claims:

1. If edge capacities are integer-valued, flows found by algorithm will be too.
   Bottleneck values will always be integers, so resulting flows are too.

2. If edge capacities are integer-valued, the algorithm will terminate.
   With integer capacities, each iteration increases flow by ≥ 1. Since the max flow is bounded (e.g., by total capacity into t), the algorithm needs $\leq$ |Max Flow| iterations.

3. If an iteration starts with a valid flow, it ends with a valid flow.
   Each iteration sends flow along a residual $s - t$ path without violating capacities or conservation of flow. So, the resulting flow is valid.

4. The first iteration starts with a valid flow.
   0 flow on all edges meets capacity and conservation of flow constraints.

Ford-Fulkerson

What can be concluded from all this?

1. Ford-Fulkerson returns a valid flow.

Claims:

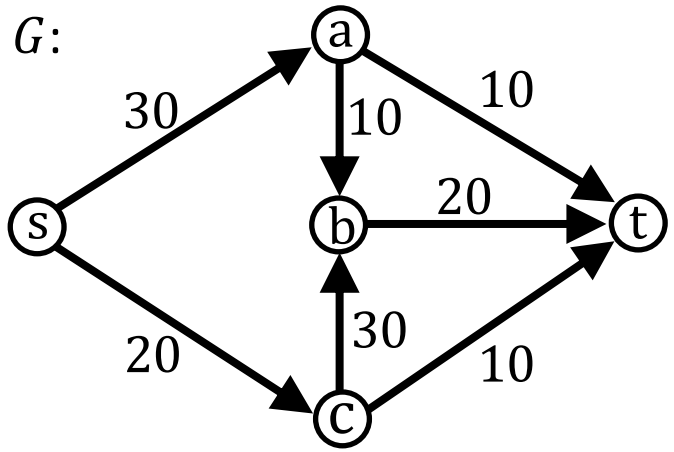
1. If edge capacities are integer-valued, flows found by algorithm will be too.
   Bottleneck values will always be integers, so resulting flows are too.

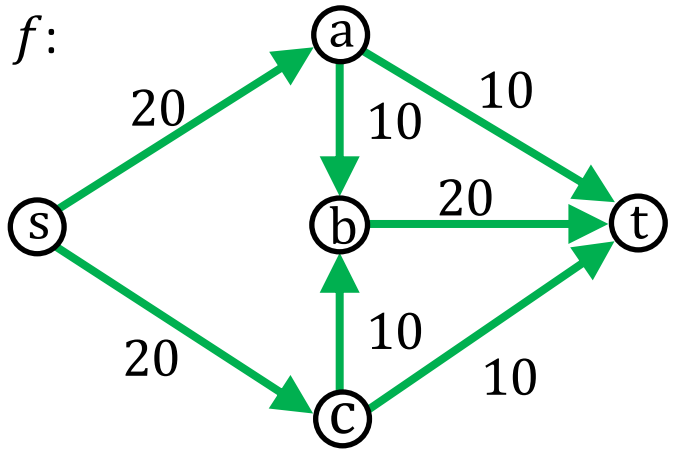
2. If edge capacities are integer-valued, the algorithm will terminate.
   With integer capacities, each iteration increases flow by ≥ 1. Since the max flow is bounded (e.g., by total capacity into t), the algorithm needs $\leq$ |Max Flow| iterations.

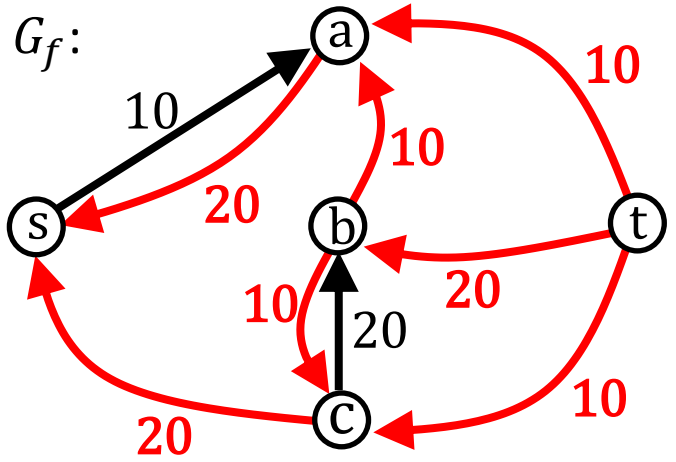
3. If an iteration starts with a valid flow, it ends with a valid flow.
   Each iteration sends flow along a residual $s - t$ path without violating capacities or conservation of flow. So, the resulting flow is valid.

4. The first iteration starts with a valid flow.
   0 flow on all edges meets capacity and conservation of flow constraints.

Ford-Fulkerson








```
Max-Flow(G)
   f(e) = 0 for all
   while s-t path in
      P = simple s-t
      f'= augment(f,
      f = f'
      Gf = Gf'
   return f
```

```
augment(f, P)
   bottleneck(P,f)
   each edge (u, v) in P
      (u, v) is a back edge
      f((v, u)) -= b
   se
      f((u, v)) += b
   return f
```

Need to show:

1. ~~Validity.~~
2. Running time.
3. Finds max flow.

Ford-Fulkerson

What can be concluded from all this?

1. Ford-Fulkerson returns a valid flow.

2. The running time is in $O(|\text{Max Flow}|)$.

Claims:

1. If edge capacities are integer-valued, flows found by algorithm will be too.

   Bottleneck values will always be integers, so resulting flows are too.

2. If edge capacities are integer-valued, the algorithm will terminate.

   With integer capacities, each iteration increases flow by ≥ 1. Since the max flow is bounded (e.g., by total capacity into t), the algorithm needs $\leq |\text{Max Flow}|$ iterations.

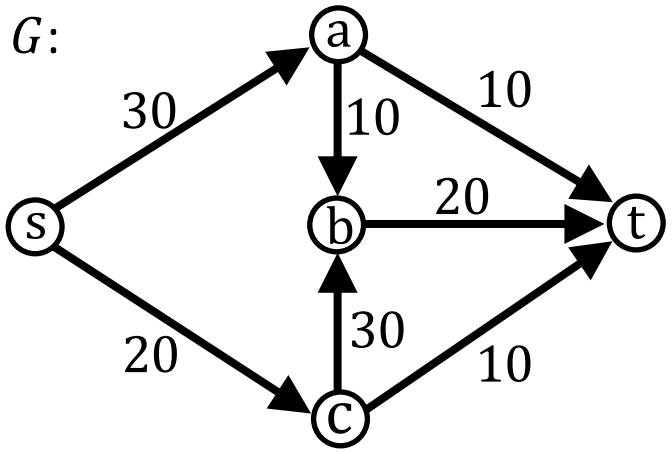
3. If an iteration starts with a valid flow, it ends with a valid flow.

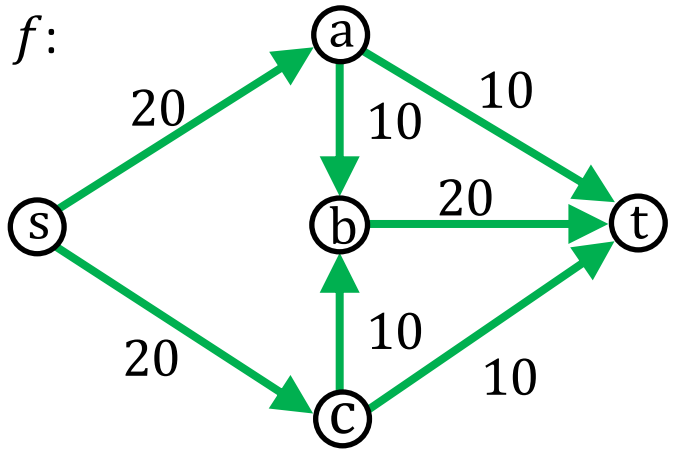
   Each iteration sends flow along a residual $s - t$ path without violating capacities or conservation of flow. So, the resulting flow is valid.

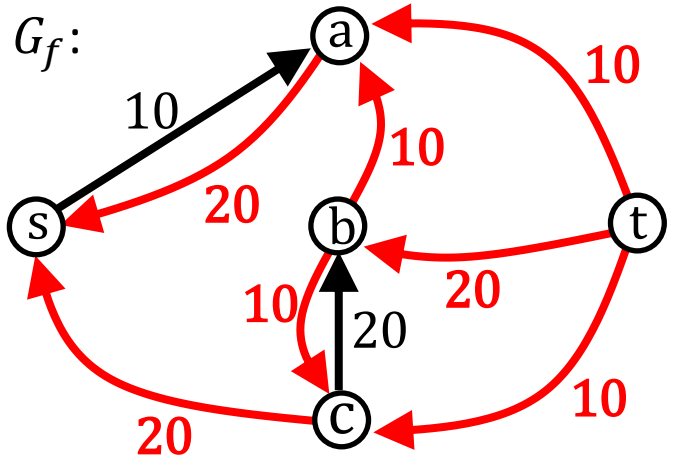
4. The first iteration starts with a valid flow.

   0 flow on all edges meets capacity and conservation of flow constraints.

Ford-Fulkerson





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Max-Flow(G)
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      f = f'
      Gf = Gf'
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augment(f, P)
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      f((v, u)) -= b
   se
      f((u, v)) += b
   return f
```

Need to show:

1. ~~Validity.~~
2. ~~Running time.~~
3. Finds max flow.

Ford-Fulkerson – Running Time

Assuming integer edge capacities:

While loop runs at most ??? times.

```
Max-Flow(G)
   f(e) = 0 for all e in G
   while s-t path in Gf exists
      P = simple s-t path in Gf
      f'= augment(f, P)
      f = f'
      Gf = Gf'
   return f
```

```
augment(f, P)
   b = bottleneck(P,f)
   for each edge (u, v) in P
      if (u, v) is a back edge
         f((v, u)) -= b
      else
         f((u, v)) += b
   return f
```

Ford-Fulkerson – Running Time

Assuming integer edge capacities:

While loop runs at most $|f_{OPT}|$ times.

```
Max-Flow(G)
   f(e) = 0 for all e in G
   while s-t path in Gf exists
      P = simple s-t path in Gf
      f'= augment(f, P)
      f = f'
      Gf = Gf'
   return f
```

```
augment(f, P)
   b = bottleneck(P,f)
   for each edge (u, v) in P
      if (u, v) is a back edge
         f((v, u)) -= b
      else
         f((u, v)) += b
   return f
```

Ford-Fulkerson – Running Time

Assuming integer edge capacities:

While loop runs at most $|f_{OPT}|$ times.

Find $s - t$ path ???

```
Max-Flow(G)
   f(e) = 0 for all e in G
   while s-t path in Gf exists
      P = simple s-t path in Gf
      f'= augment(f, P)
      f = f'
      Gf = Gf'
   return f
```

```
augment(f, P)
   b = bottleneck(P,f)
   for each edge (u, v) in P
      if (u, v) is a back edge
         f((v, u)) -= b
      else
         f((u, v)) += b
   return f
```

Ford-Fulkerson – Running Time

Assuming integer edge capacities:

While loop runs at most $|f_{OPT}|$ times.

Find $s - t$ path (BFS/DFS): $O(|E| + |V|)$

```
Max-Flow(G)
   f(e) = 0 for all e in G
   while s-t path in Gf exists
      P = simple s-t path in Gf
      f'= augment(f, P)
      f = f'
      Gf = Gf'
   return f
```

```
augment(f, P)
   b = bottleneck(P,f)
   for each edge (u, v) in P
      if (u, v) is a back edge
         f((v, u)) -= b
      else
         f((u, v)) += b
   return f
```

Ford-Fulkerson – Running Time

Assuming integer edge capacities:

While loop runs at most $|f_{OPT}|$ times.

Find $s - t$ path (BFS/DFS): $O(|E| + |V|)$

`augment(f, P)` ???

```
Max-Flow(G)
   f(e) = 0 for all e in G
   while s-t path in Gf exists
      P = simple s-t path in Gf
      f'= augment(f, P)
      f = f'
      Gf = Gf'
   return f
```

```
augment(f, P)
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   for each edge (u, v) in P
      if (u, v) is a back edge
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         f((u, v)) += b
   return f
```

Ford-Fulkerson – Running Time

Assuming integer edge capacities:

While loop runs at most $|f_{OPT}|$ times.

Find $s - t$ path (BFS/DFS): $O(|E| + |V|)$

`augment(f, P)` just traverses edges: $O(|V|)$

```
Max-Flow(G)
   f(e) = 0 for all e in G
   while s-t path in Gf exists
      P = simple s-t path in Gf
      f'= augment(f, P)
      f = f'
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   b = bottleneck(P,f)
   for each edge (u, v) in P
      if (u, v) is a back edge
         f((v, u)) -= b
      else
         f((u, v)) += b
   return f
```

Ford-Fulkerson – Running Time

Assuming integer edge capacities:

- While loop runs at most $|f_{OPT}|$ times.
- Find $s - t$ path (BFS/DFS): $O(|E| + |V|)$
- `augment(f, P)` just traverses edges: $O(|V|)$
- Update G_f ???

```
Max-Flow(G)
   f(e) = 0 for all e in G
   while s-t path in Gf exists
      P = simple s-t path in Gf
      f'= augment(f, P)
      f = f'
      Gf = Gf'
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Ford-Fulkerson – Running Time

Assuming integer edge capacities:

- While loop runs at most $|f_{OPT}|$ times.
- Find $s - t$ path (BFS/DFS): $O(|E| + |V|)$
- `augment(f, P)` just traverses edges: $O(|V|)$
- Update G_f (for each $e \in$ P, make e and $e' \in G_f$): $O(|E|)$

```
Max-Flow(G)
   f(e) = 0 for all e in G
   while s-t path in Gf exists
      P = simple s-t path in Gf
      f'= augment(f, P)
      f = f'
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   return f
```

Ford-Fulkerson – Running Time

Assuming integer edge capacities:

- While loop runs at most $|f_{OPT}|$ times.
- Find $s - t$ path (BFS/DFS): $O(|E| + |V|)$
- `augment(f, P)` just traverses edges: $O(|V|)$
- Update G_f (for each $e \in$ P, make e and $e' \in G_f$): $O(|E|)$

Total = $O(|f_{OPT}|\ 2\ (|E| + |V|)) = O(|E| \cdot |f_{OPT}|)$

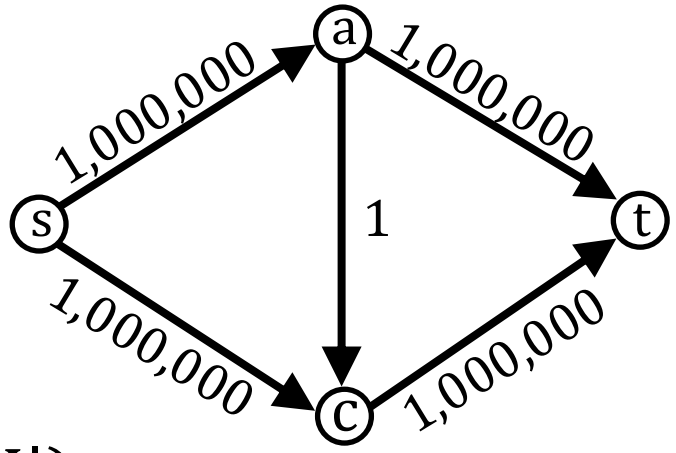
Good?

Bad?

```
Max-Flow(G)
   f(e) = 0 for all e in G
   while s-t path in Gf exists
      P = simple s-t path in Gf
      f'= augment(f, P)
      f = f'
      Gf = Gf'
   return f
```

```
augment(f, P)
   b = bottleneck(P,f)
   for each edge (u, v) in P
      if (u, v) is a back edge
         f((v, u)) -= b
      else
         f((u, v)) += b
   return f
```

Ford-Fulkerson – Running Time



Assuming integer edge capacities:

- While loop runs at most $|f_{OPT}|$ times.
- Find $s - t$ path (BFS/DFS): $O(|E| + |V|)$
- `augment(f, P)` just traverses edges: $O(|V|)$
- Update G_f (for each $e \in$ P, make e and $e' \in G_f$): $O(|E|)$

Total = $O(|f_{OPT}|\ 2\ (|E| + |V|)) = O(|E| \cdot |f_{OPT}|)$

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Max-Flow(G)
   f(e) = 0 for all e in G
   while s-t path in Gf exists
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      f = f'
      Gf = Gf'
   return f
```

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augment(f, P)
   b = bottleneck(P,f)
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Ford-Fulkerson

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augment(f, P)
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   for each edge (u, v) in P
      if (u, v) is a back edge
         f((v, u)) -= b
      else
         f((u, v)) += b
   return f
```

Edmonds-Karp

```
Max-Flow(G)
   f(e) = 0 for all e in G
   while s-t path in Gf exists
      P = shortest s-t path in Gf
      f' = augment(f, P)
      f = f'
      Gf = Gf'
   return f
```

```
augment(f, P)
   b = bottleneck(P,f)
   for each edge (u, v) in P
      if (u, v) is a back edge
         f((v, u)) -= b
      else
         f((u, v)) += b
   return f
```

Shortest = smallest number of edges.

Edmonds-Karp – Running Time

```
Max-Flow(G)
   f(e) = 0 for all e in G
   while s-t path in Gf exists
      P = shortest s-t path in Gf
      f'= augment(f, P)
      f = f'
      Gf = Gf'
   return f
```

```
augment(f, P)
   b = bottleneck(P,f)
   for each edge (u, v) in P
      if (u, v) is a back edge
         f((v, u)) -= b
      else
         f((u, v)) += b
   return f
```

Edmonds-Karp – Running Time

Claim 1: The distance of the shortest path from s to any v in the residual graph is non-decreasing over the iterations of the algorithm.

```
Max-Flow(G)
   f(e) = 0 for all e in G
   while s-t path in Gf exists
      P = shortest s-t path in Gf
      f'= augment(f, P)
      f = f'
      Gf = Gf'
   return f
```

```
augment(f, P)
   b = bottleneck(P,f)
   for each edge (u, v) in P
      if (u, v) is a back edge
         f((v, u)) -= b
      else
         f((u, v)) += b
   return f
```

Edmonds-Karp – Running Time

Claim 1: The distance of the shortest path from s to any v in the residual graph is non-decreasing over the iterations of the algorithm.

Intuition: Paths change in the residual graph as we remove edges by filling them up or adding (back) edges by using forward edges for the first time.

```
Max-Flow(G)
   f(e) = 0 for all e in G
   while s-t path in Gf exists
      P = shortest s-t path in Gf
      f'= augment(f, P)
      f = f'
      Gf = Gf'
   return f
```

```
augment(f, P)
   b = bottleneck(P,f)
   for each edge (u, v) in P
      if (u, v) is a back edge
         f((v, u)) -= b
      else
         f((u, v)) += b
   return f
```

Edmonds-Karp – Running Time

Claim 1: The distance of the shortest path from s to any v in the residual graph is non-decreasing over the iterations of the algorithm.

Intuition: Adding flow to a residual graph can only make paths longer by removing (saturated) edges or adding back edges, which to be able to use, first requires reaching its head (at least as far as before) plus one extra hop.

```
Max-Flow(G)
   f(e) = 0 for all e in G
   while s-t path in Gf exists
      P = shortest s-t path in Gf
      f'= augment(f, P)
      f = f'
      Gf = Gf'
   return f
```

```
augment(f, P)
   b = bottleneck(P,f)
   for each edge (u, v) in P
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   return f
```

Edmonds-Karp – Running Time

Claim 1: The distance of the shortest path from s to any v in the residual graph is non-decreasing over the iterations of the algorithm.

Claim 2: Each edge can be the bottleneck for a path at most $O(|V|)$ times.

```
Max-Flow(G)
   f(e) = 0 for all e in G
   while s-t path in Gf exists
      P = shortest s-t path in Gf
      f'= augment(f, P)
      f = f'
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         f((u, v)) += b
   return f
```

Edmonds-Karp – Running Time

Claim 1: The distance of the shortest path from s to any v in the residual graph is non-decreasing over the iterations of the algorithm.

Claim 2: Each edge can be the bottleneck for a path at most $O(|V|)$ times.

Intuition: An edge cannot be a bottleneck again without the shortest distance increasing, which can happen at most $|V| - 1$ times.

```
Max-Flow(G)
   f(e) = 0 for all e in G
   while s-t path in Gf exists
      P = shortest s-t path in Gf
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```
augment(f, P)
   b = bottleneck(P,f)
   for each edge (u, v) in P
      if (u, v) is a back edge
         f((v, u)) -= b
      else
         f((u, v)) += b
   return f
```

Edmonds-Karp – Running Time

Claim 1: The distance of the shortest path from s to any v in the residual graph is non-decreasing over the iterations of the algorithm.

Claim 2: Each edge can be the bottleneck for a path at most $O(|V|)$ times.

Thus, each iteration needs a bottleneck + at most $|E| * O(|V|)$ bottlenecks

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Max-Flow(G)
   f(e) = 0 for all e in G
   while s-t path in Gf exists
      P = shortest s-t path in Gf
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      f = f'
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```

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augment(f, P)
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Edmonds-Karp – Running Time

Claim 1: The distance of the shortest path from s to any v in the residual graph is non-decreasing over the iterations of the algorithm.

Claim 2: Each edge can be the bottleneck for a path at most $O(|V|)$ times.

Thus, each iteration needs a bottleneck + at most $|E| * O(|V|)$ bottlenecks

$\Rightarrow$ at most $O(|E||V|)$ iterations $\Rightarrow$

```
Max-Flow(G)
   f(e) = 0 for all e in G
   while s-t path in Gf exists
      P = shortest s-t path in Gf
      f'= augment(f, P)
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augment(f, P)
   b = bottleneck(P,f)
   for each edge (u, v) in P
      if (u, v) is a back edge
         f((v, u)) -= b
      else
         f((u, v)) += b
   return f
```

Edmonds-Karp – Running Time

Claim 1: The distance of the shortest path from s to any v in the residual graph is non-decreasing over the iterations of the algorithm.

Claim 2: Each edge can be the bottleneck for a path at most $O(|V|)$ times.

Thus, each iteration needs a bottleneck + at most $|E| * O(|V|)$ bottlenecks

$\Rightarrow$ at most $O(|E||V|)$ iterations $\Rightarrow O(|E|^2|V|)$ time total.

```
Max-Flow(G)
   f(e) = 0 for all e in G
   while s-t path in Gf exists
      P = shortest s-t path in Gf
      f'= augment(f, P)
      f = f'
      Gf = Gf'
   return f
```

```
augment(f, P)
   b = bottleneck(P,f)
   for each edge (u, v) in P
      if (u, v) is a back edge
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Running Time:

$O(|E| \cdot |f_{OPT}|)$ [Ford-Fulkerson, 1956]

$O(|V||E|^2)$ [Edmonds-Karp, 1972]

ge

```
    f'= augment(f, P)                    f((v, u)) -= b
    f = f'                             else
    Gf = Gf'                             f((u, v)) += b
return f                             return f
```

Edmonds-Karp – Running Time

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- $O(|V||E|)$ [Orlin, 2013]
- $O\left(|E|^{1+o(1)}\right)$ [Chen et al., 2022]

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$\in \boldsymbol{O}(\textbf{Max Flow})$

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