Flow Networks
CSCI 432

Optimality

```
Max-Flow(G)
   f(e) = 0 for all e in G
   while s-t path in G_f exists
      P = simple s-t path in G_f
      f'= augment(f, P)
      f = f'
      G_f = G_f'
   return f
```

```
augment(f, P)
   b = bottleneck(P,f)
   for each edge (u, v) in P
      if (u, v) is a back edge
         f((v, u)) -= b
      else
         f((u, v)) += b
   return f
```

Need to show:

1. ~~Validity.~~
2. ~~Running time.~~
3. Finds max flow.

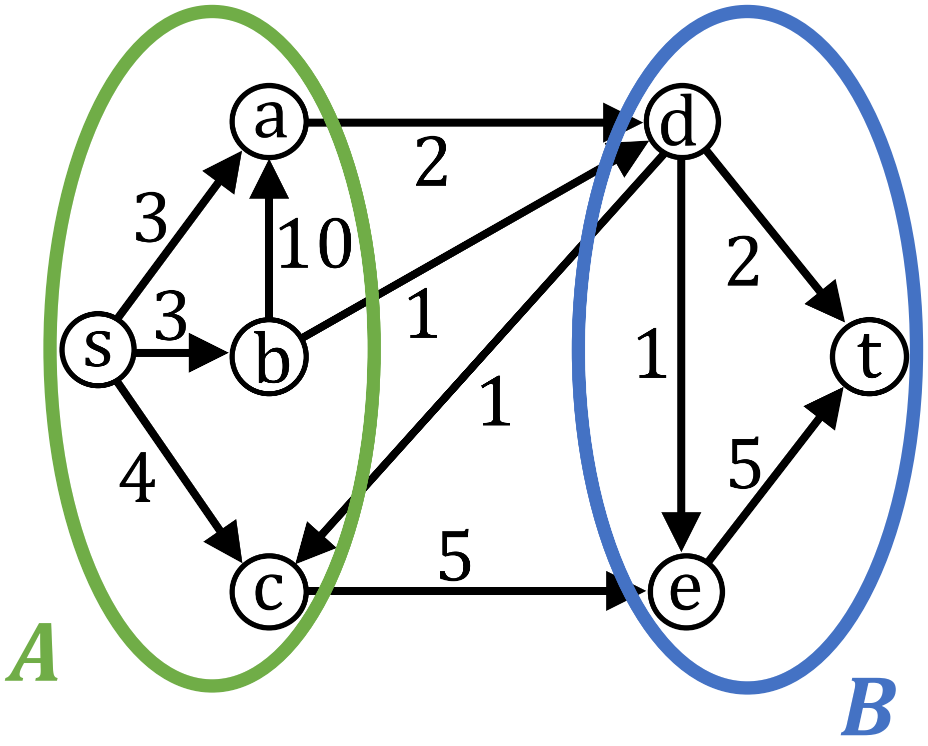
Optimality

Theorem: The flow returned by the Ford-Fulkerson algorithm is a maximum flow.

Proof: ...

Optimality

<u>Definitions:</u> Suppose G is a flow network and nodes in G are divided into two sets, $\boldsymbol{A}$ and $\boldsymbol{B}$, such that $s \in A$ and $t \in B$. We call (A, B) an $s - t$ *cut*. The *capacity* of the cut, $c(A, B)$, is the sum of capacities of all edges out of A.

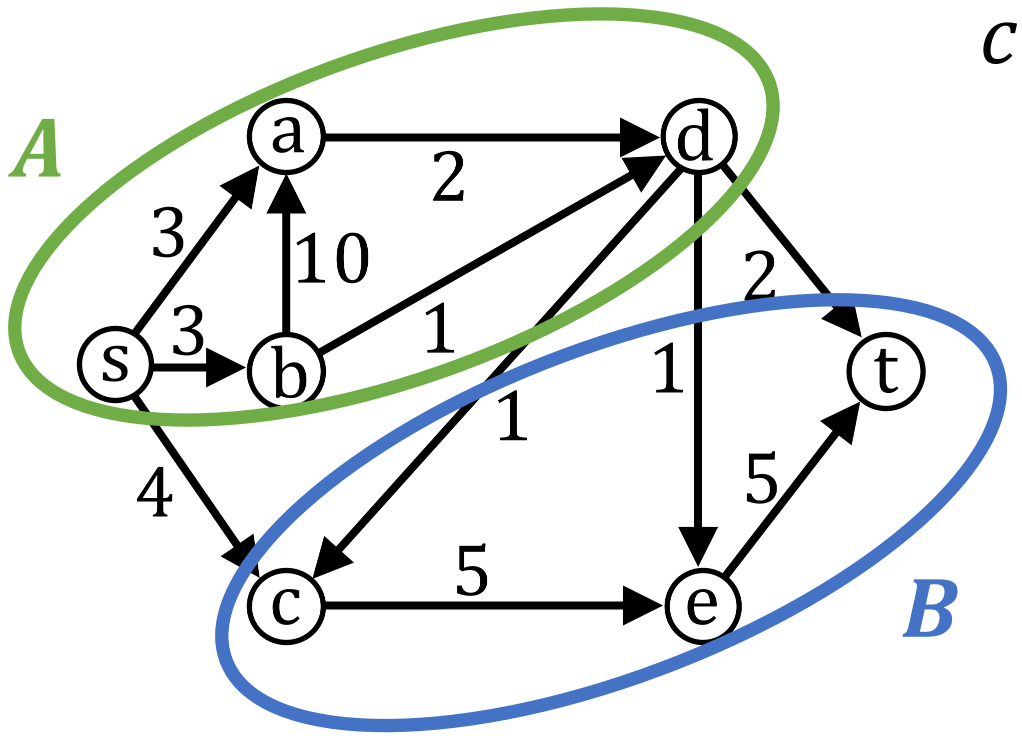


$$c(A, B) = 8$$

Optimality

<u>Definitions:</u> Suppose G is a flow network and nodes in G are divided into two sets, $\boldsymbol{A}$ and $\boldsymbol{B}$, such that $s \in A$ and $t \in B$. We call (A, B) an $s - t$ *cut*. The *capacity* of the cut, $c(A, B)$, is the sum of capacities of all edges out of A.

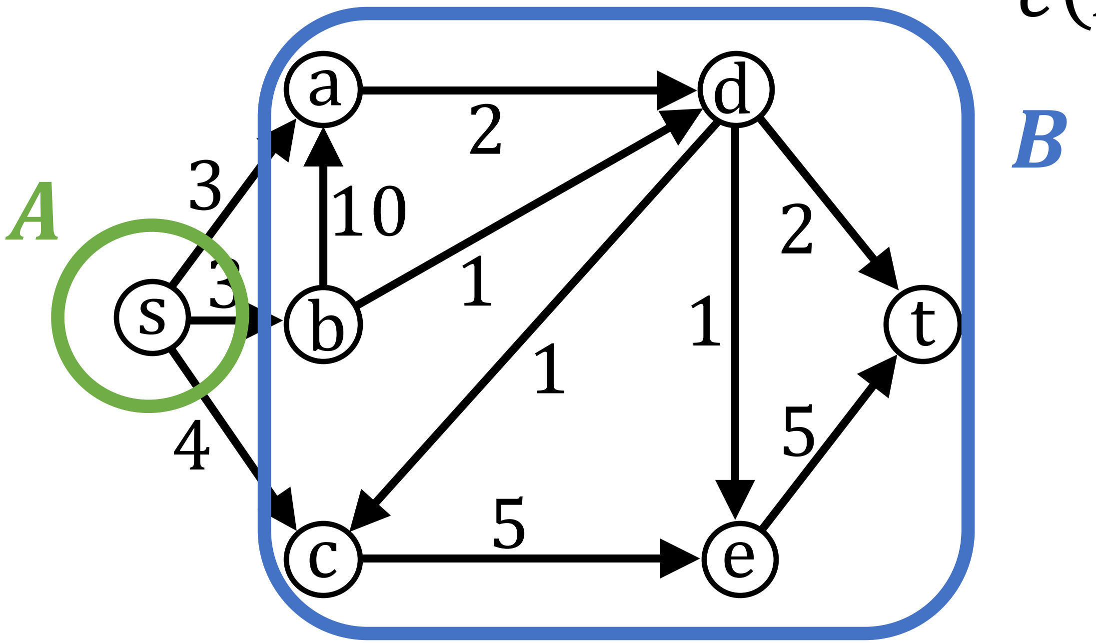
$$c(A, B) = 8$$



Optimality

<u>Definitions:</u> Suppose G is a flow network and nodes in G are divided into two sets, $\boldsymbol{A}$ and $\boldsymbol{B}$, such that $s \in A$ and $t \in B$. We call (A, B) an $s - t$ *cut*. The *capacity* of the cut, $c(A, B)$, is the sum of capacities of all edges out of A.

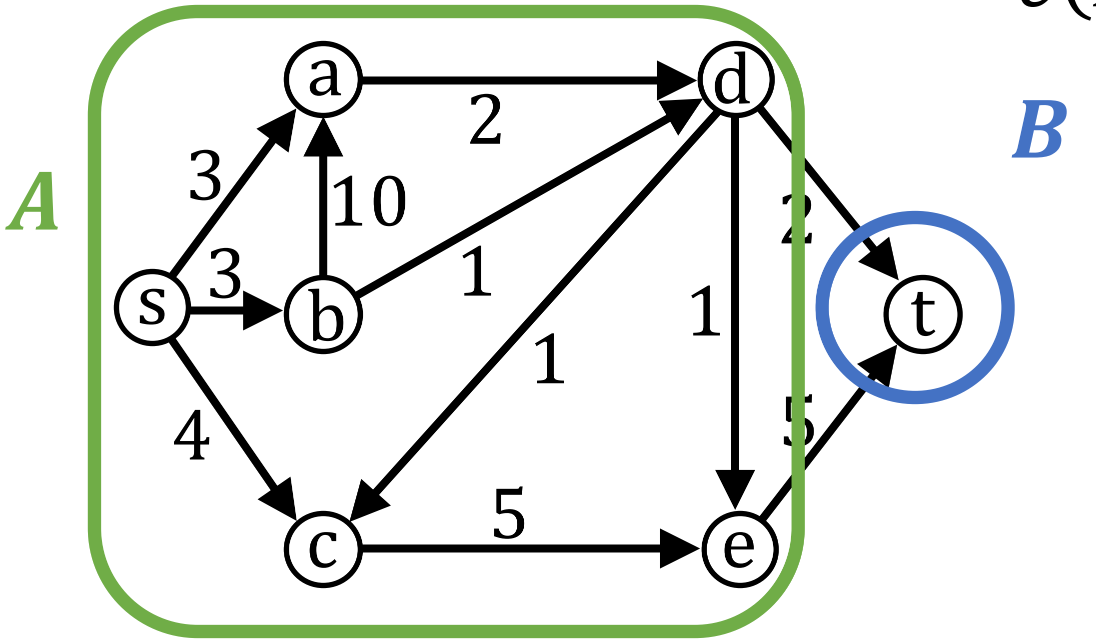
$$c(A, B) = 10$$



Optimality

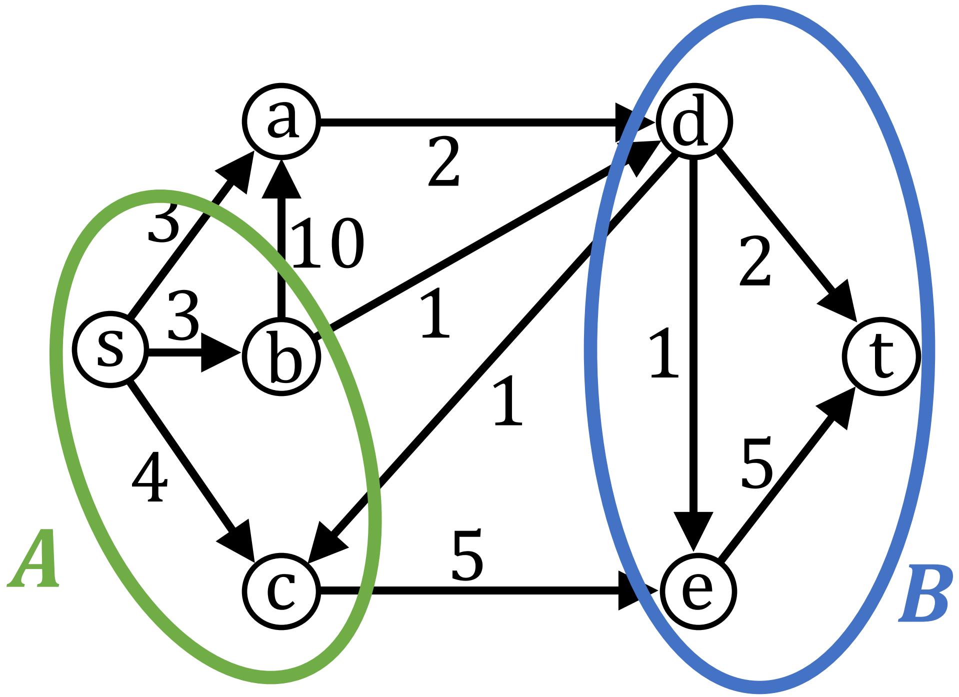
Definitions: Suppose G is a flow network and nodes in G are divided into two sets, $\boldsymbol{A}$ and $\boldsymbol{B}$, such that $s \in A$ and $t \in B$. We call (A, B) an $s - t$ *cut*. The *capacity* of the cut, $c(A, B)$, is the sum of capacities of all edges out of A.

$$c(A, B) = 7$$



Optimality

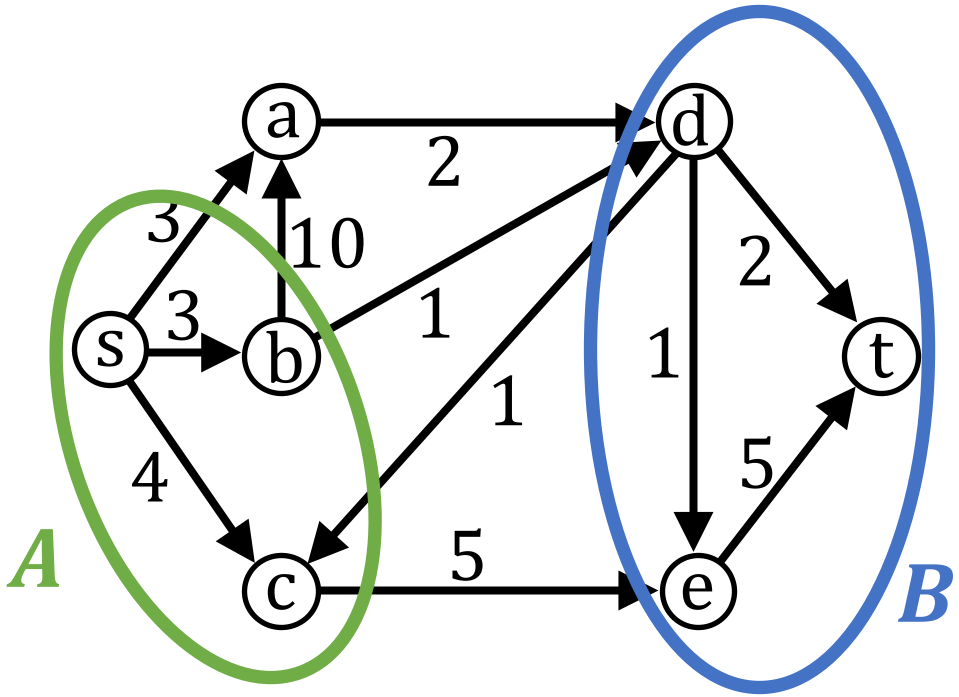
<u>Definitions:</u> Suppose G is a flow network and nodes in G are divided into two sets, $\boldsymbol{A}$ and $\boldsymbol{B}$, such that $s \in A$ and $t \in B$. We call (A, B) an $s - t$ *cut*. The *capacity* of the cut, $c(A, B)$, is the sum of capacities of all edges out of A.



$$c(A, B) = ??$$

Optimality

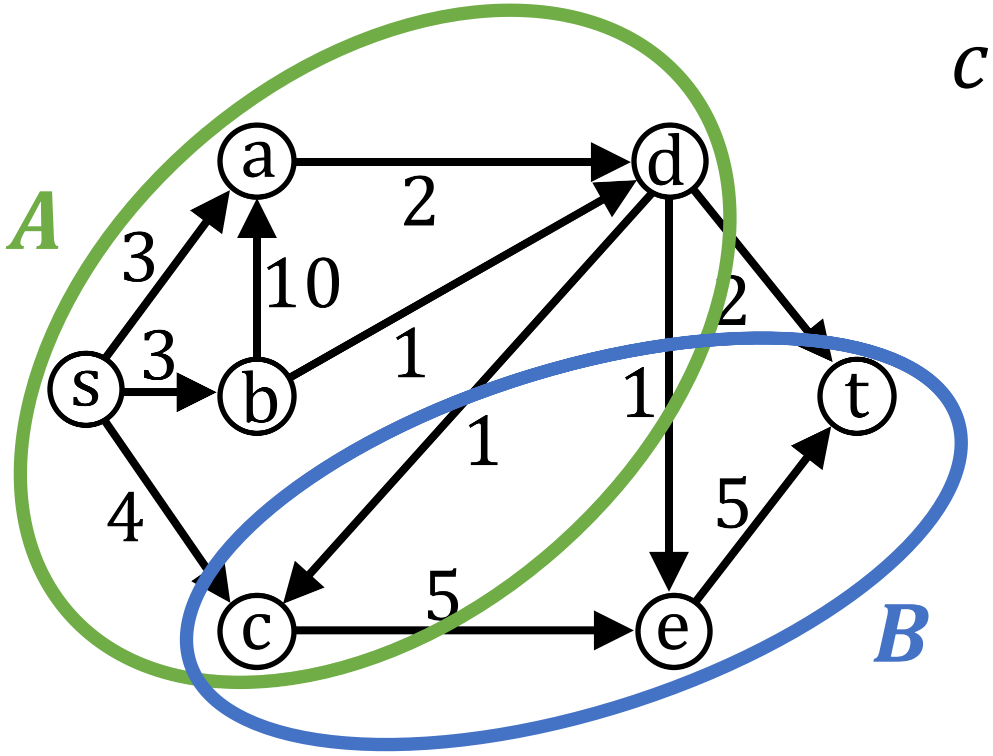
Definitions: Suppose G is a flow network and nodes in G are divided into two sets, ***A*** and ***B***, such that $s \in A$ and $t \in B$. We call (A, B) an $s - t$ *cut*. The *capacity* of the cut, $c(A, B)$, is the sum of capacities of all edges out of A.



Invalid cut! Every vertex needs to be is in one of the sets!

Optimality

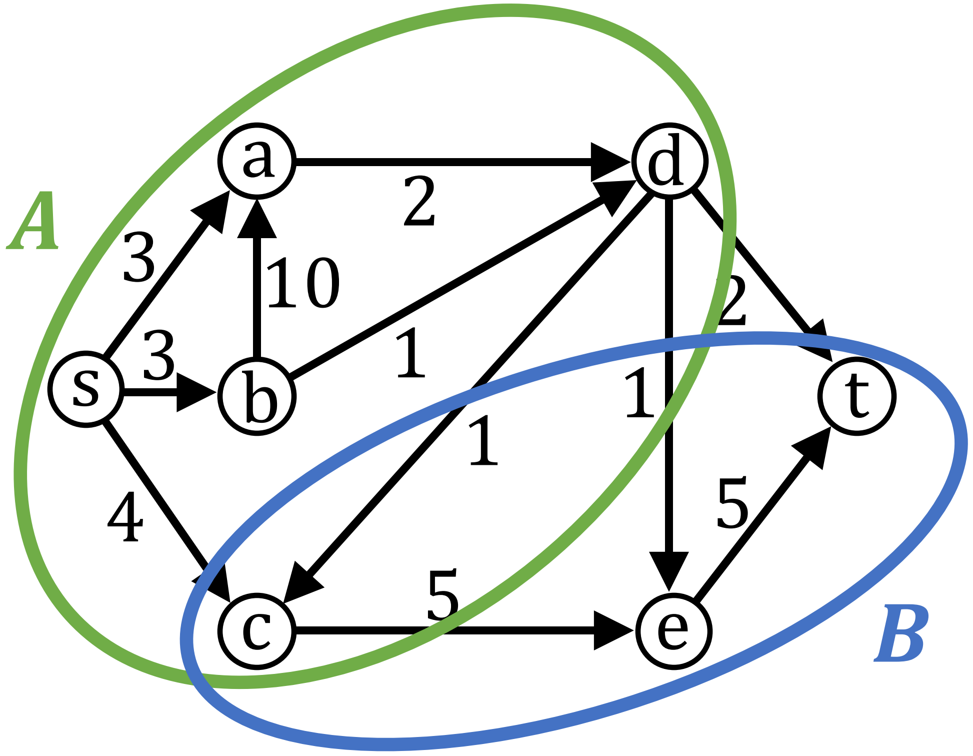
Definitions: Suppose G is a flow network and nodes in G are divided into two sets, ***A*** and ***B***, such that $s \in A$ and $t \in B$. We call (A, B) an $s - t$ *cut*. The *capacity* of the cut, $c(A, B)$, is the sum of capacities of all edges out of A.



$$c(A, B) = ??$$

Optimality

<u>Definitions:</u> Suppose G is a flow network and nodes in G are divided into two sets, $\boldsymbol{A}$ and $\boldsymbol{B}$, such that $s \in A$ and $t \in B$. We call (A, B) an $s - t$ *cut*. The *capacity* of the cut, $c(A, B)$, is the sum of capacities of all edges out of A.

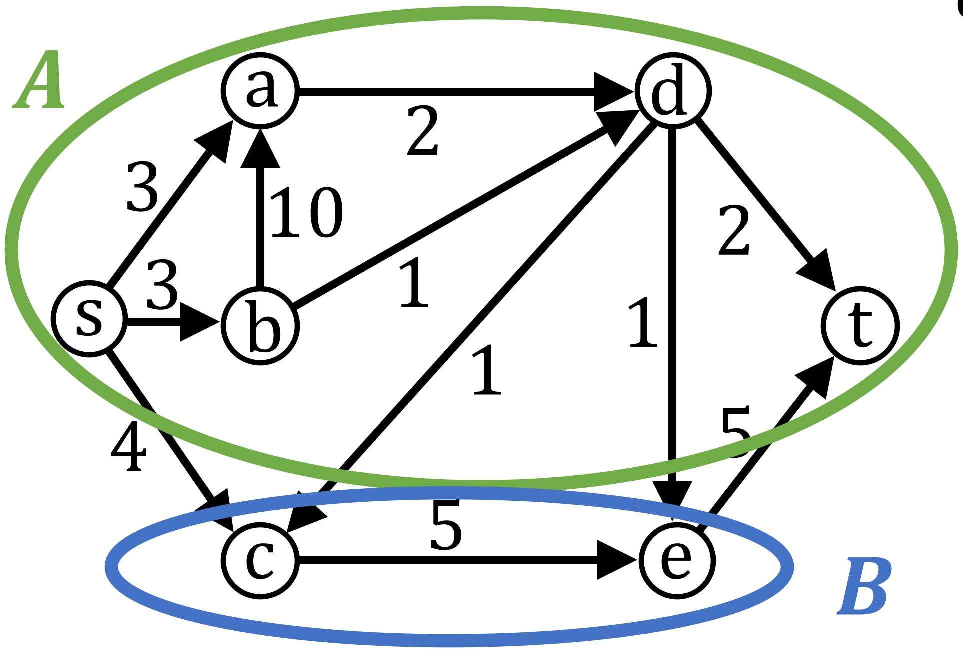


Invalid cut! Every vertex needs to be in <u>exactly</u> one of the sets!

Optimality

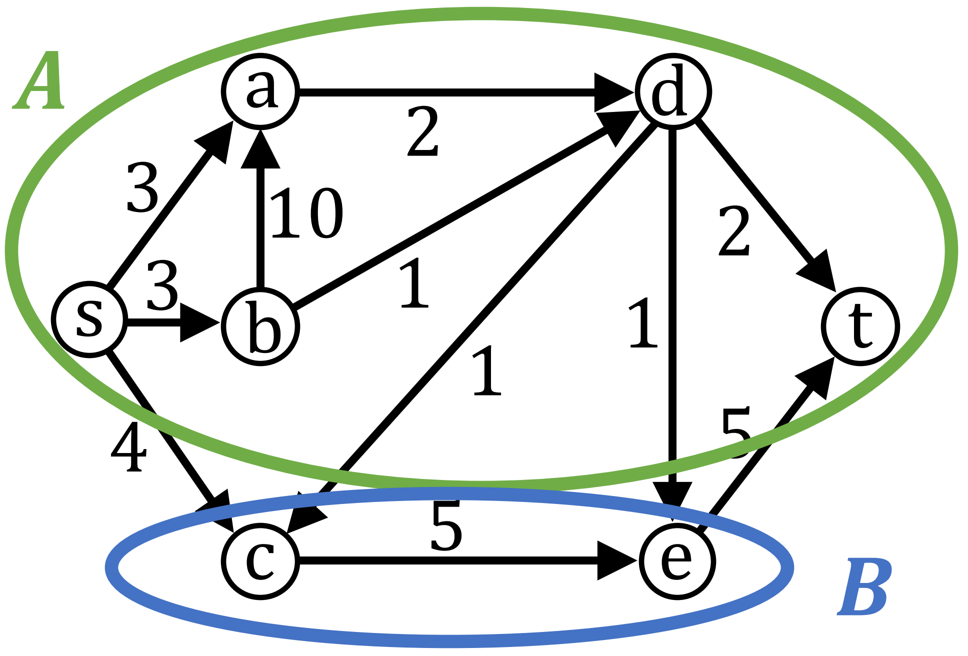
Definitions: Suppose G is a flow network and nodes in G are divided into two sets, $\boldsymbol{A}$ and $\boldsymbol{B}$, such that $s \in A$ and $t \in B$. We call (A, B) an $s - t$ *cut*. The *capacity* of the cut, $c(A, B)$, is the sum of capacities of all edges out of A.

$$c(A, B) = ??$$



Optimality

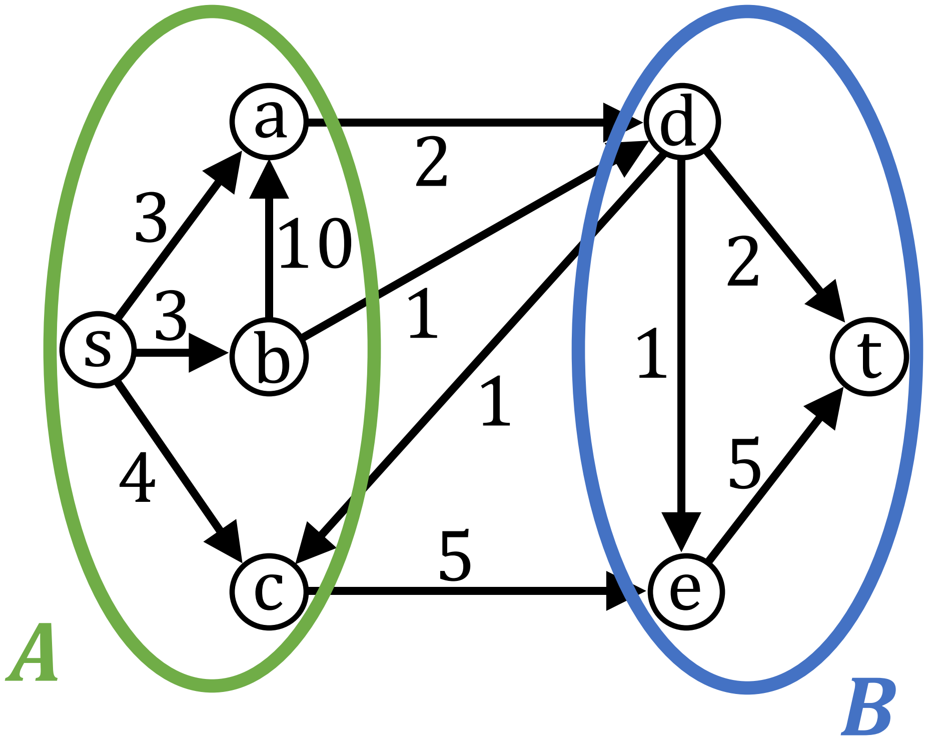
<u>Definitions:</u> Suppose G is a flow network and nodes in G are divided into two sets, $\boldsymbol{A}$ and $\boldsymbol{B}$, such that $s \in A$ and $t \in B$. We call (A, B) an $s - t$ *cut*. The *capacity* of the cut, $c(A, B)$, is the sum of capacities of all edges out of A.



Invalid $s - t$ cut! s and t need to be in different sets!

Optimality

Definitions: Suppose G is a flow network and nodes in G are divided into two sets, $\boldsymbol{A}$ and $\boldsymbol{B}$, such that $s \in A$ and $t \in B$. We call (A, B) an $s - t$ *cut*. The *capacity* of the cut, $c(A, B)$, is the sum of capacities of all edges out of A.

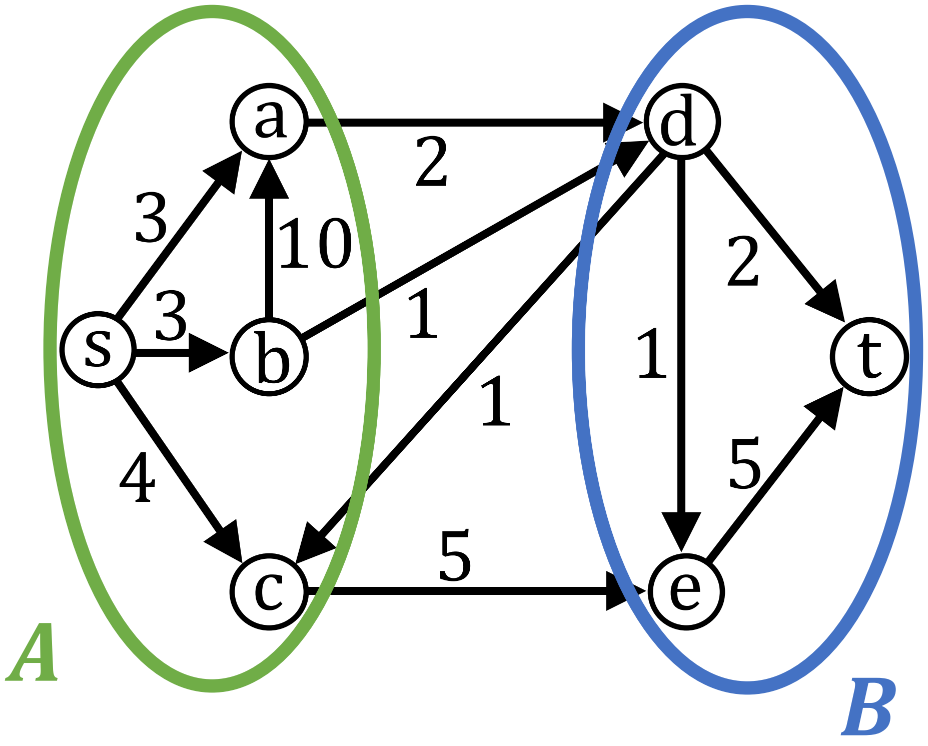


$$c(A, B) = 8$$

Game Plan:

Optimality

Definitions: Suppose G is a flow network and nodes in G are divided into two sets, $\boldsymbol{A}$ and $\boldsymbol{B}$, such that $s \in A$ and $t \in B$. We call (A, B) an $s - t$ *cut*. The *capacity* of the cut, $c(A, B)$, is the sum of capacities of all edges out of A.



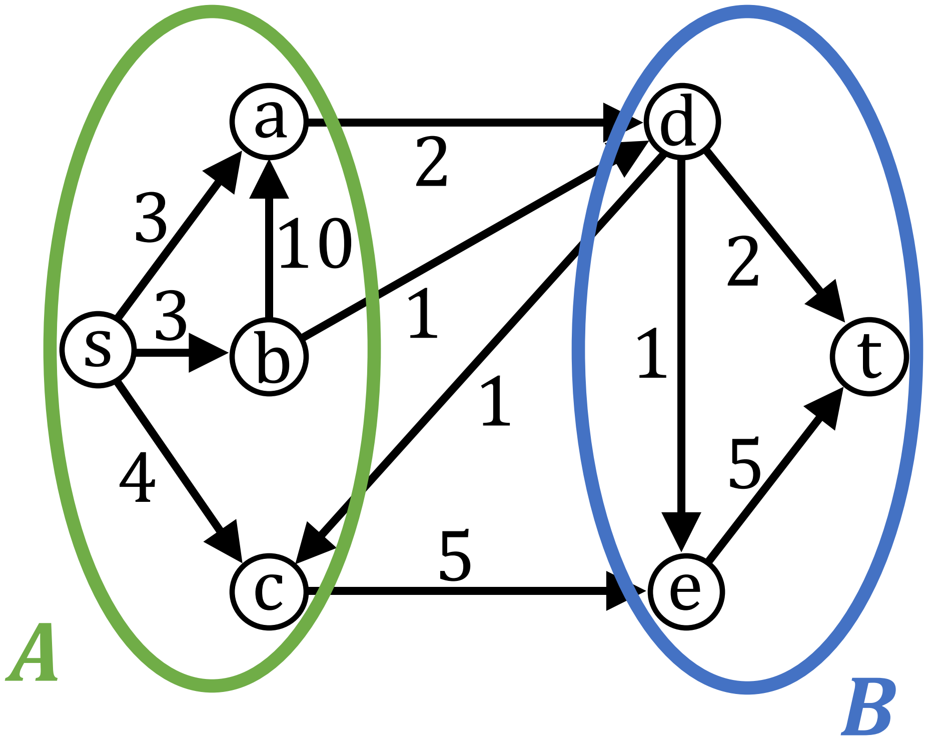
$$c(A, B) = 8$$

Game Plan:

1. Show the value of every flow is $\leq$ capacity of every cut.

Optimality

<u>Definitions:</u> Suppose G is a flow network and nodes in G are divided into two sets, $\boldsymbol{A}$ and $\boldsymbol{B}$, such that $s \in A$ and $t \in B$. We call (A, B) an $s - t$ *cut*. The *capacity* of the cut, $c(A, B)$, is the sum of capacities of all edges out of A.



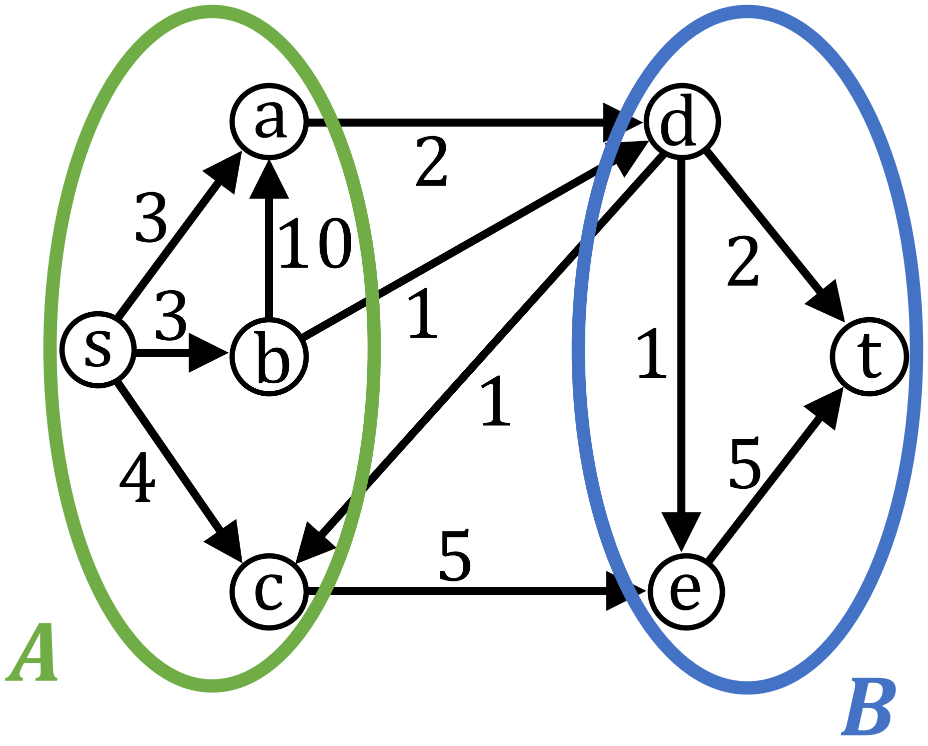
$$c(A, B) = 8$$

Game Plan:

1. Show the value of every flow is ≤ capacity of every cut.

Optimality

Definitions: Suppose G is a flow network and nodes in G are divided into two sets, $\boldsymbol{A}$ and $\boldsymbol{B}$, such that $s \in A$ and $t \in B$. We call (A, B) an $s - t$ *cut*. The *capacity* of the cut, $c(A, B)$, is the sum of capacities of all edges out of A.



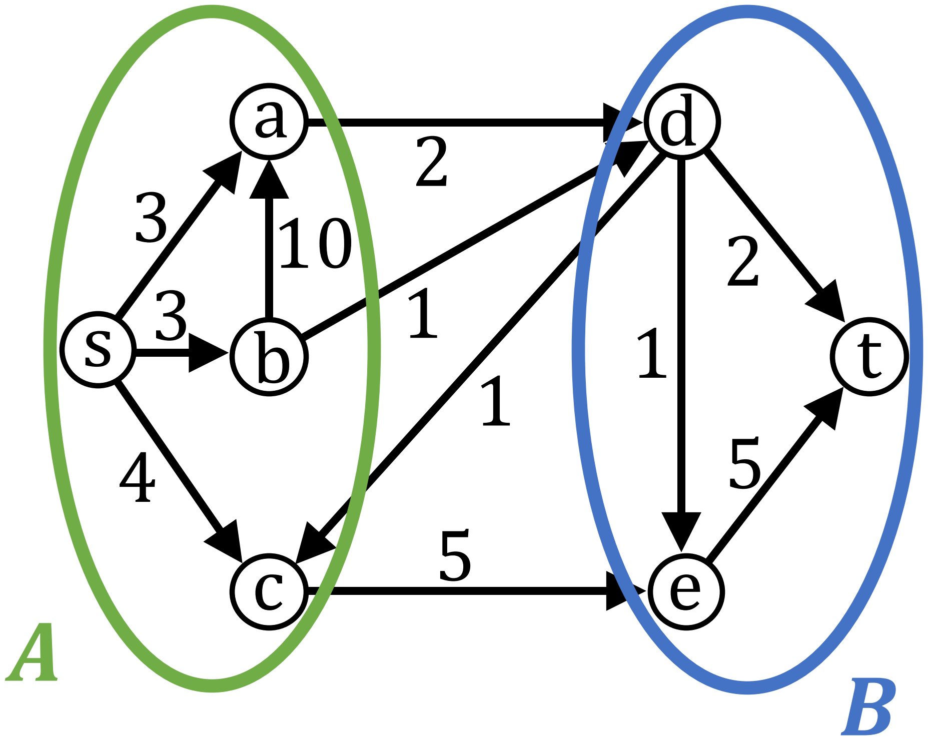
$$c(A, B) = 8$$

Game Plan:

1. Show the value of every flow is ≤ capacity of every cut.

Optimality

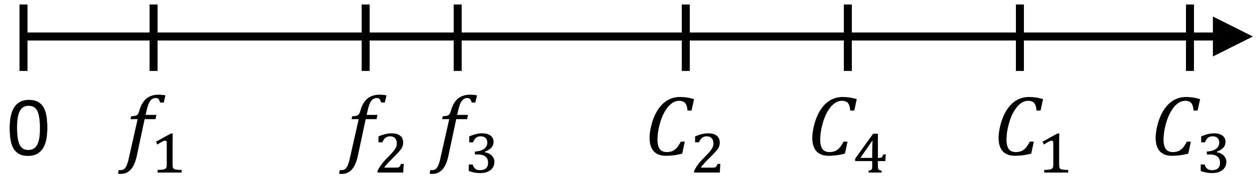
Definitions: Suppose G is a flow network and nodes in G are divided into two sets, $\boldsymbol{A}$ and $\boldsymbol{B}$, such that $s \in A$ and $t \in B$. We call (A, B) an $s - t$ *cut*. The *capacity* of the cut, $c(A, B)$, is the sum of capacities of all edges out of A.



$$c(A, B) = 8$$

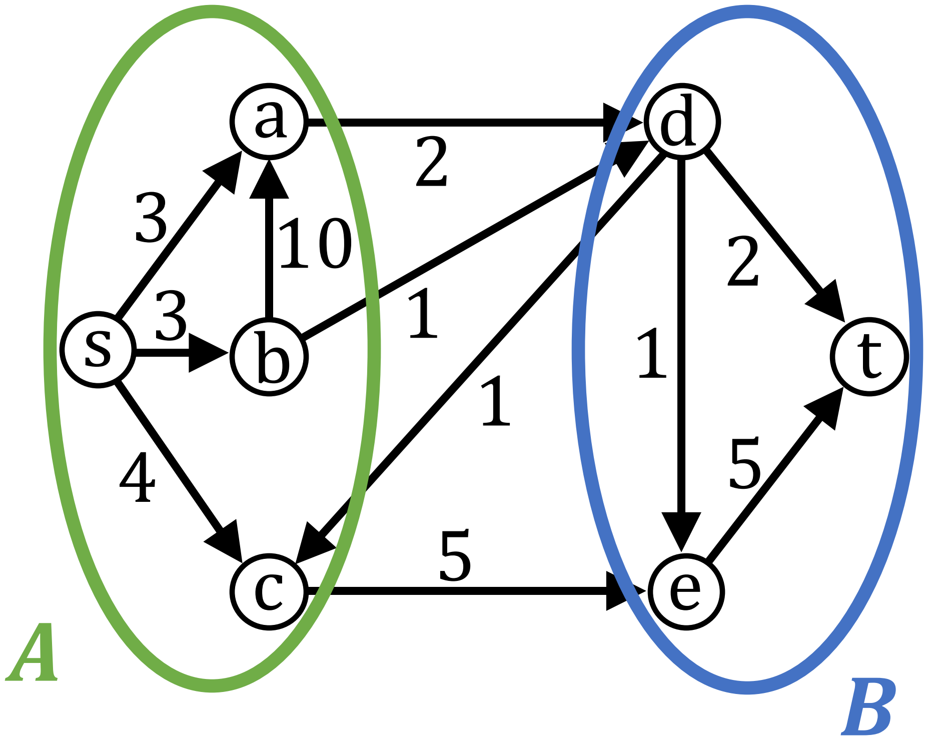
Game Plan:

1. Show the value of every flow is ≤ capacity of every cut.



Optimality

<u>Definitions:</u> Suppose G is a flow network and nodes in G are divided into two sets, $\boldsymbol{A}$ and $\boldsymbol{B}$, such that $s \in A$ and $t \in B$. We call (A, B) an $s - t$ *cut*. The *capacity* of the cut, $c(A, B)$, is the sum of capacities of all edges out of A.

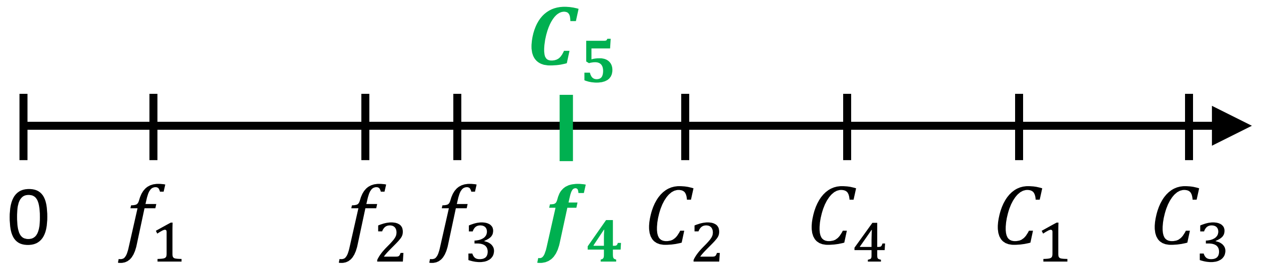


$c(A, B) = 8$

Consequence: If we find some flow whose value equals the capacity of some cut, it must be the optimal flow.

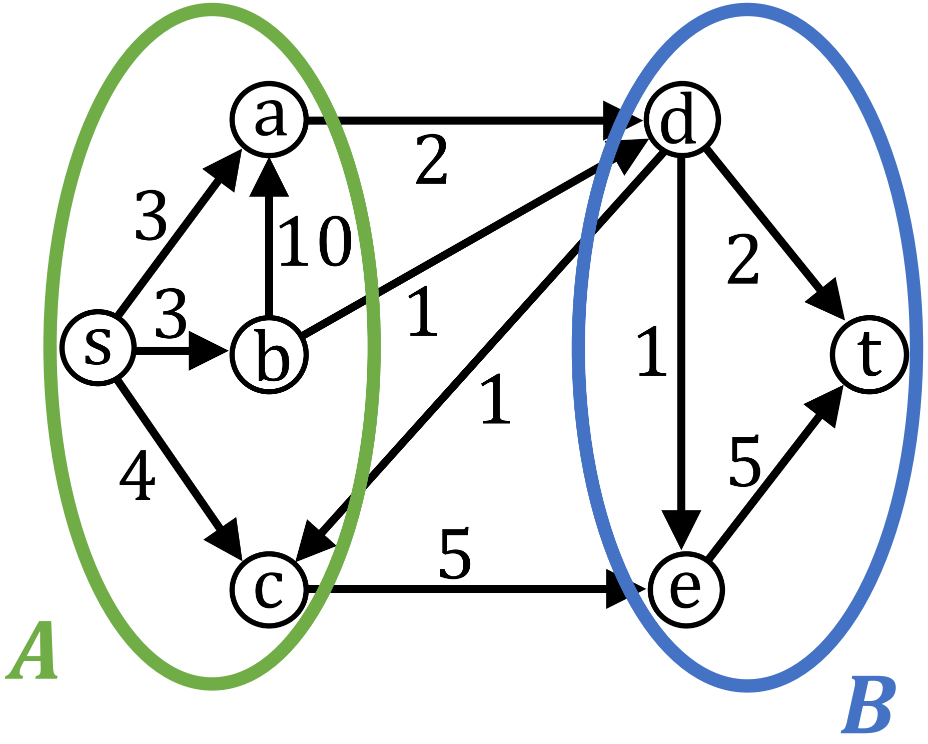
Game Plan:

1. Show the value of every flow is ≤ capacity of every cut.



Optimality

<u>Definitions:</u> Suppose G is a flow network and nodes in G are divided into two sets, $\boldsymbol{A}$ and $\boldsymbol{B}$, such that $s \in A$ and $t \in B$. We call (A, B) an $s - t$ *cut*. The *capacity* of the cut, $c(A, B)$, is the sum of capacities of all edges out of A.



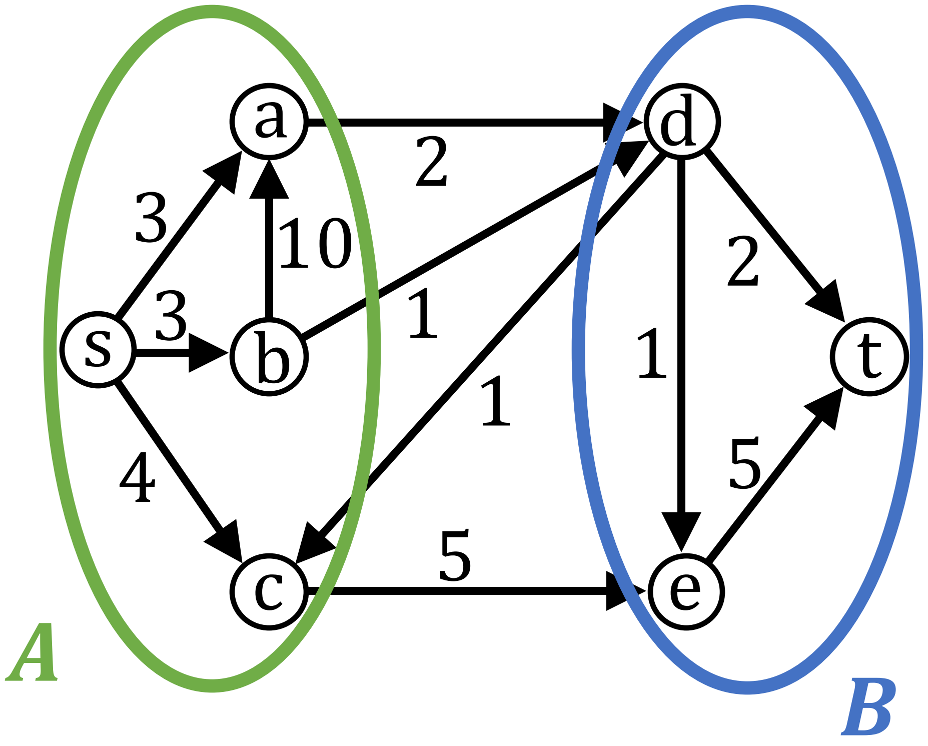
$$c(A, B) = 8$$

Game Plan:

1. Show the value of every flow is ≤ capacity of every cut.
2. Given a flow where there are no $s - t$ paths left in the residual graph, there is a specific cut whose capacity = flow value.

Optimality

<u>Definitions:</u> Suppose G is a flow network and nodes in G are divided into two sets, $\boldsymbol{A}$ and $\boldsymbol{B}$, such that $s \in A$ and $t \in B$. We call (A, B) an $s - t$ *cut*. The *capacity* of the cut, $c(A, B)$, is the sum of capacities of all edges out of A.



$$c(A, B) = 8$$

Game Plan:

1. Show the value of every flow is $\leq$ capacity of every cut.
2. Given a flow where there are no $s - t$ paths left in the residual graph, there is a specific cut whose capacity = flow value.

Consequence: The algorithm is optimal

Optimality

Theorem 1: Let G be a flow network, (A, B) be an $s - t$ cut, and f be an $s - t$ flow. Then, $|f| = \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{in}(A)} f(e)$.

Proof:

Edges that enter the set A

Edges that leave the set A

Optimality

This relates arbitrary $s-t$ flows to arbitrary $s-t$ cuts

<u>Theorem 1:</u> Let G be a flow network, (A, B) be an $s-t$ cut, and f be an $s-t$ flow. Then, $|f| = \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{in}(A)} f(e)$.

<u>Proof:</u>

Edges that enter the set A

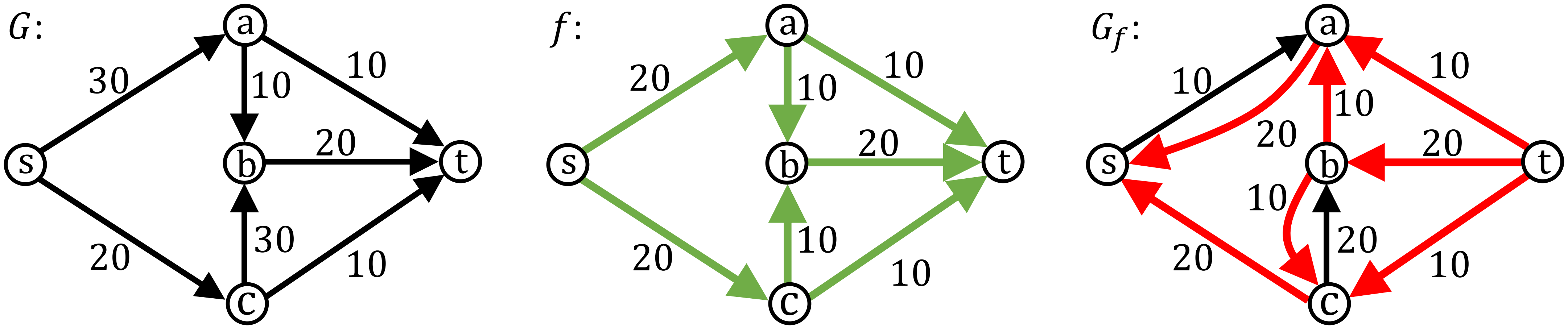
Edges that leave the set A

Optimality

<u>Theorem 1:</u> Let G be a flow network, (A, B) be an $s - t$ cut, and f be an $s - t$ flow. Then, $|f| = \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{in}(A)} f(e)$.

<u>Proof:</u>

$|f| = 40$

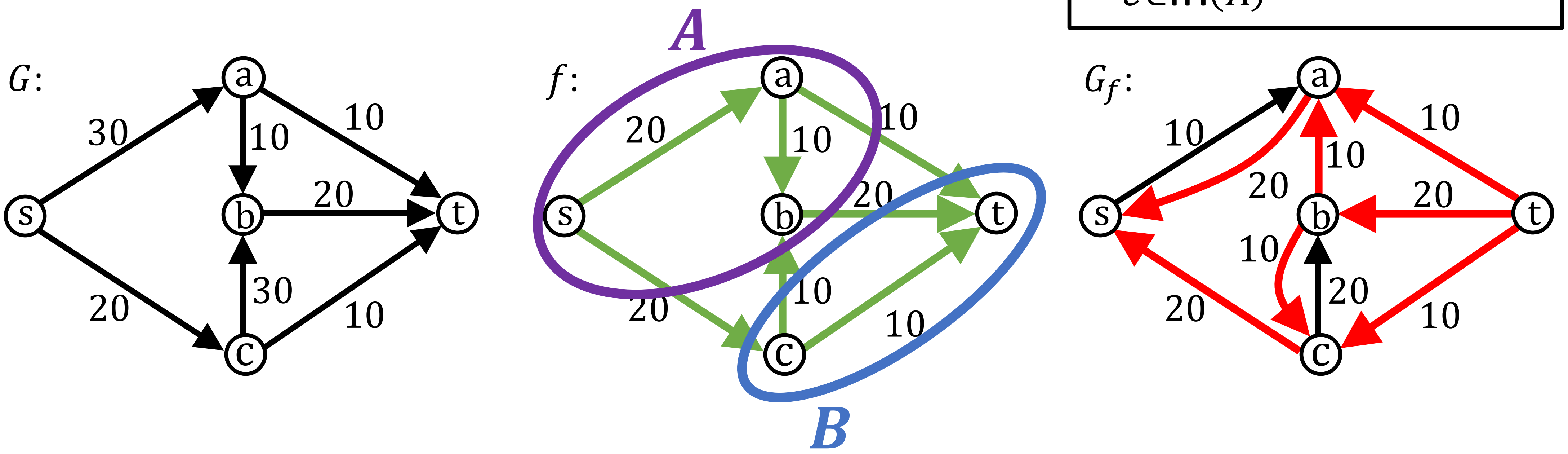


Optimality

<u>Theorem 1:</u> Let G be a flow network, (A, B) be an $s-t$ cut, and f be an $s-t$ flow. Then, $|f| = \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{in}(A)} f(e)$.

<u>Proof:</u>

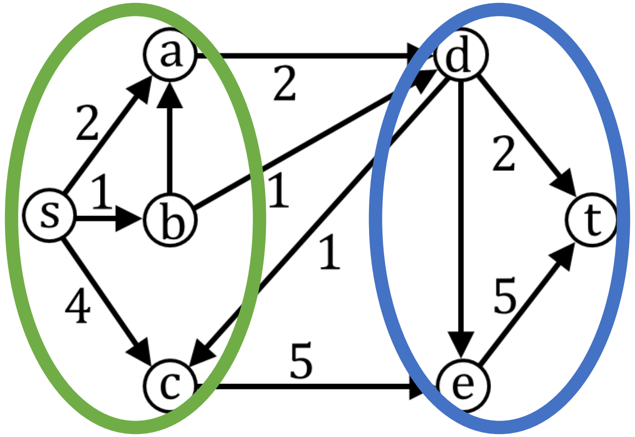
$|f| = 40$
$\sum_{e \in \text{out}(A)} f(e) = 50$
$\sum_{e \in \text{in}(A)} f(e) = 10$



Optimality

<u>Theorem 1:</u> Let G be a flow network, (A, B) be an $s-t$ cut, and f be an $s-t$ flow. Then, $|f| = \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{in}(A)} f(e)$.

<u>Proof:</u> Let $f^{out}(v) = \sum_{e \in \text{out}(v)} f(e)$ and $f^{in}(v) = \sum_{e \in \text{in}(v)} f(e)$.

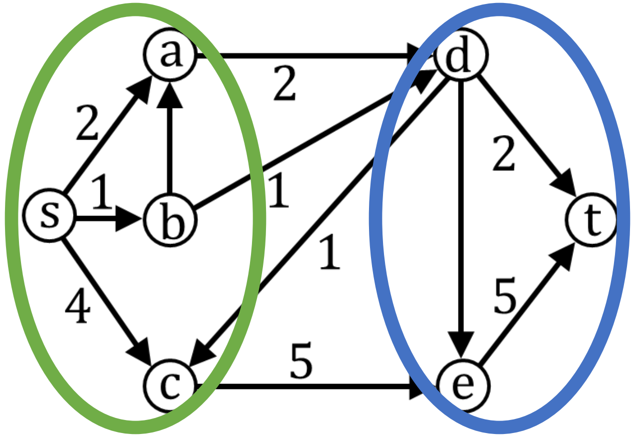


Optimality

<u>Theorem 1:</u> Let G be a flow network, (A, B) be an $s - t$ cut, and f be an $s - t$ flow. Then, $|f| = \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{in}(A)} f(e)$.

<u>Proof:</u> Let $f^{out}(v) = \sum_{e \in \text{out}(v)} f(e)$ and $f^{in}(v) = \sum_{e \in \text{in}(v)} f(e)$.

Then, $\forall v \in A, v \neq s, f^{out}(v) - f^{in}(v) = ?$

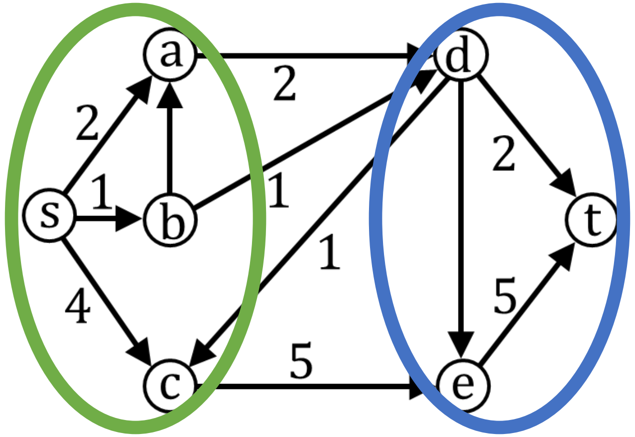


Optimality

<u>Theorem 1:</u> Let G be a flow network, (A, B) be an $s - t$ cut, and f be an $s - t$ flow. Then, $|f| = \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{in}(A)} f(e)$.

<u>Proof:</u> Let $f^{out}(v) = \sum_{e \in \text{out}(v)} f(e)$ and $f^{in}(v) = \sum_{e \in \text{in}(v)} f(e)$.

Then, $\forall v \in A, v \neq s, f^{out}(v) - f^{in}(v) = 0$ (by conservation of flow).



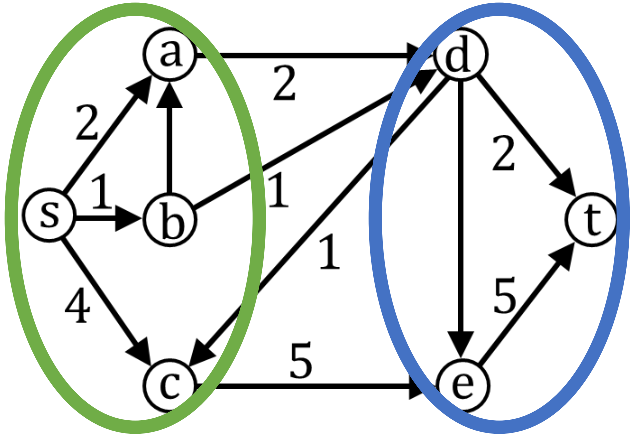
Optimality

Theorem 1: Let G be a flow network, (A, B) be an $s-t$ cut, and f be an $s-t$ flow. Then, $|f| = \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{in}(A)} f(e)$.

Proof: Let $f^{out}(v) = \sum_{e \in \text{out}(v)} f(e)$ and $f^{in}(v) = \sum_{e \in \text{in}(v)} f(e)$.

Then, $\forall v \in A, v \neq s, f^{out}(v) - f^{in}(v) = 0$ (by conservation of flow).

By definition, $|f| = f^{out}(s)$.

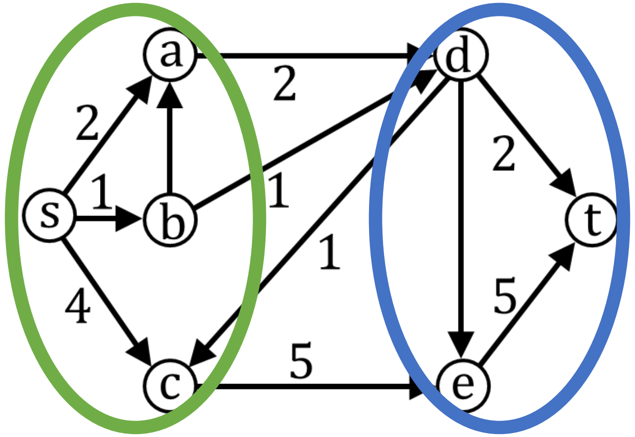


Optimality

Theorem 1: Let G be a flow network, (A, B) be an $s - t$ cut, and f be an $s - t$ flow.
Then, $|f| = \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{in}(A)} f(e)$.

Proof: Let $f^{out}(v) = \sum_{e \in \text{out}(v)} f(e)$ and $f^{in}(v) = \sum_{e \in \text{in}(v)} f(e)$.
Then, $\forall v \in A, v \neq s, f^{out}(v) - f^{in}(v) = 0$ (by conservation of flow).
By definition, $|f| = f^{out}(s)$.

$\Longrightarrow |f| = f^{out}(s) - f^{in}(s)$ (since $f^{in}(s) = 0$)



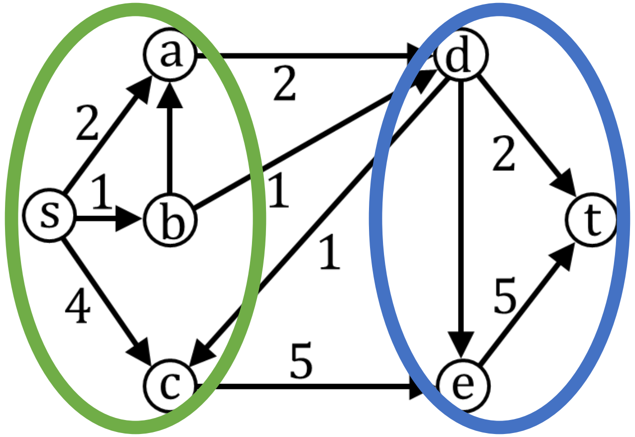
Optimality

Theorem 1: Let G be a flow network, (A, B) be an $s - t$ cut, and f be an $s - t$ flow.
Then, $|f| = \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{in}(A)} f(e)$.

Proof: Let $f^{out}(v) = \sum_{e \in \text{out}(v)} f(e)$ and $f^{in}(v) = \sum_{e \in \text{in}(v)} f(e)$.
Then, $\forall v \in A, v \neq s, f^{out}(v) - f^{in}(v) = 0$ (by conservation of flow).
By definition, $|f| = f^{out}(s)$.

$\Longrightarrow |f| = f^{out}(s) - f^{in}(s)$ (since $f^{in}(s) = 0$)

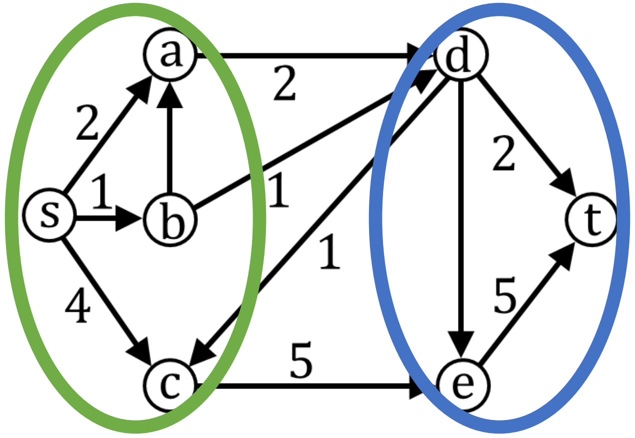
$\Longrightarrow |f| = \sum_{v \in A}(f^{out}(v) - f^{in}(v))$ (Only $\neq 0$ when $v = s$).



Optimality

Theorem 1: Let G be a flow network, (A, B) be an $s-t$ cut, and f be an $s-t$ flow. Then, $|f| = \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{in}(A)} f(e)$.

Proof: Let $f^{out}(v) = \sum_{e \in \text{out}(v)} f(e)$ and $f^{in}(v) = \sum_{e \in \text{in}(v)} f(e)$.
Then, $|f| = \sum_{v \in A}(f^{out}(v) - f^{in}(v))$ (Only $\neq 0$ when $v = s$).



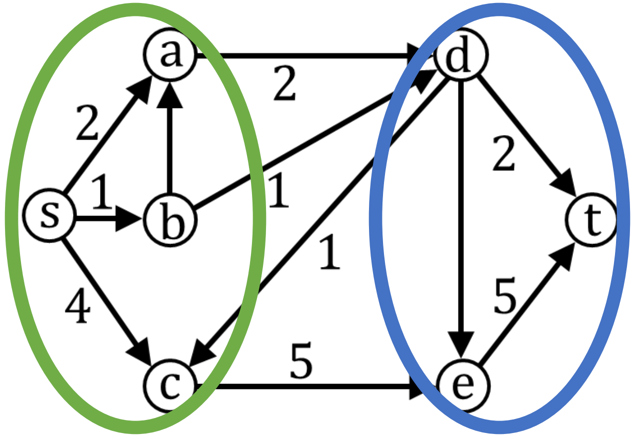
Optimality

Theorem 1: Let G be a flow network, (A, B) be an $s - t$ cut, and f be an $s - t$ flow. Then, $|f| = \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{in}(A)} f(e)$.

Proof: Let $f^{out}(v) = \sum_{e \in \text{out}(v)} f(e)$ and $f^{in}(v) = \sum_{e \in \text{in}(v)} f(e)$.

Then, $|f| = \sum_{v \in A}(f^{out}(v) - f^{in}(v))$ (Only $\neq 0$ when $v = s$).

Need to translate vertices in A into edges leaving A.



Optimality

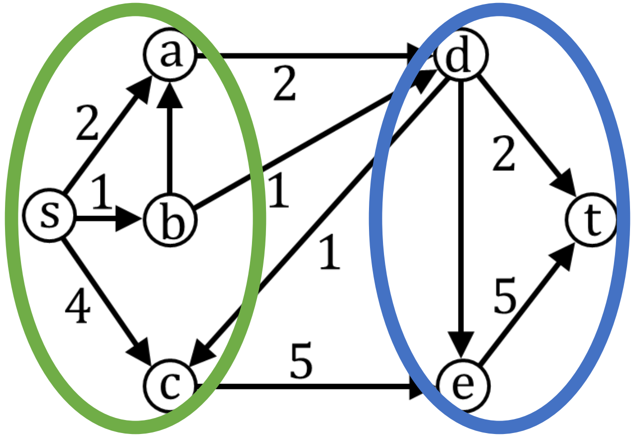
Theorem 1: Let G be a flow network, (A, B) be an $s - t$ cut, and f be an $s - t$ flow. Then, $|f| = \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{in}(A)} f(e)$.

Proof: Let $f^{out}(v) = \sum_{e \in \text{out}(v)} f(e)$ and $f^{in}(v) = \sum_{e \in \text{in}(v)} f(e)$.

Then, $|f| = \sum_{v \in A}(f^{out}(v) - f^{in}(v))$ (Only $\neq 0$ when $v = s$).

$\forall e = (u, v) \in E$:

1. $u \in A, v \in A$ (edge is inside A)
2. $u \notin A, v \notin A$ (edge is outside A)
3. $u \in A, v \notin A$ (edge leaves A)
4. $u \notin A, v \in A$ (edge enters A)



Optimality

Theorem 1: Let G be a flow network, (A, B) be an $s - t$ cut, and f be an $s - t$ flow. Then, $|f| = \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{in}(A)} f(e)$.

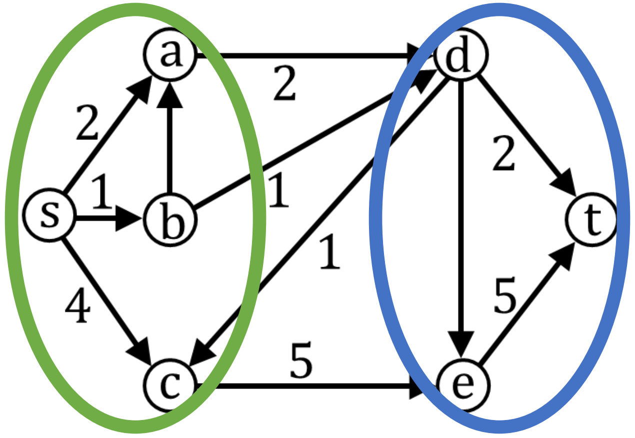
Proof: Let $f^{out}(v) = \sum_{e \in \text{out}(v)} f(e)$ and $f^{in}(v) = \sum_{e \in \text{in}(v)} f(e)$.

Then, $|f| = \sum_{v \in A}(f^{out}(v) - f^{in}(v))$ (Only $\neq 0$ when $v = s$).

$\forall e = (u, v) \in E$:

1. $u \in A, v \in A \rightarrow f(e)$ cancels out.
2. $u \notin A, v \notin A$ (edge is outside A)
3. $u \in A, v \notin A$ (edge leaves A)
4. $u \notin A, v \in A$ (edge enters A)

(In some $v \in A$ and out another)



Optimality

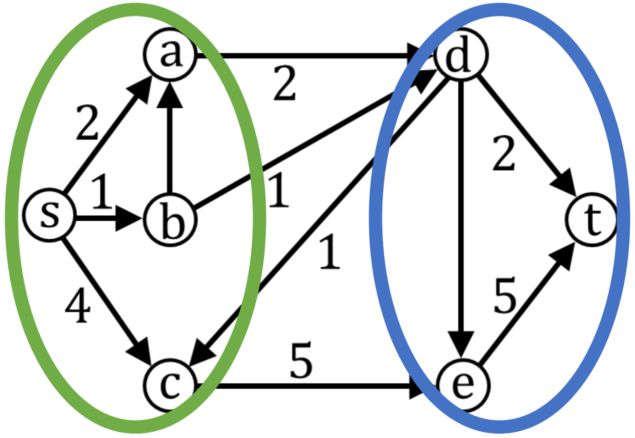
Theorem 1: Let G be a flow network, (A, B) be an $s - t$ cut, and f be an $s - t$ flow. Then, $|f| = \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{in}(A)} f(e)$.

Proof: Let $f^{out}(v) = \sum_{e \in \text{out}(v)} f(e)$ and $f^{in}(v) = \sum_{e \in \text{in}(v)} f(e)$.

Then, $|f| = \sum_{v \in A}(f^{out}(v) - f^{in}(v))$ (Only $\neq 0$ when $v = s$).

$\forall e = (u, v) \in E$:

1. ~~$u \in A, v \in A \to f(e)$ cancels out.~~ (In some $v \in A$ and out another)
2. $u \notin A, v \notin A$ **(edge is outside A)**
3. $u \in A, v \notin A$ **(edge leaves A)**
4. $u \notin A, v \in A$ **(edge enters A)**



Optimality

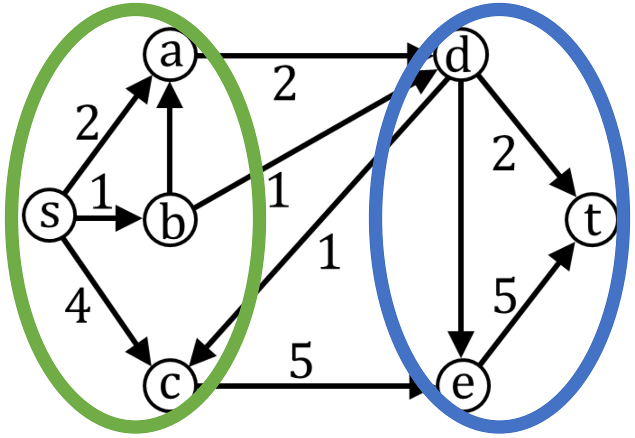
Theorem 1: Let G be a flow network, (A, B) be an $s - t$ cut, and f be an $s - t$ flow. Then, $|f| = \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{in}(A)} f(e)$.

Proof: Let $f^{out}(v) = \sum_{e \in \text{out}(v)} f(e)$ and $f^{in}(v) = \sum_{e \in \text{in}(v)} f(e)$.

Then, $|f| = \sum_{v \in A} (f^{out}(v) - f^{in}(v))$ (Only $\neq 0$ when $v = s$).

$\forall e = (u, v) \in E$:

1. ~~$u \in A, v \in A \rightarrow f(e)$ cancels out.~~
2. ~~$u \notin A, v \notin A \rightarrow f(e)$ does not appear at all.~~
3. $u \in A, v \notin A$ **(edge leaves A)**
4. $u \notin A, v \in A$ **(edge enters A)**



Optimality

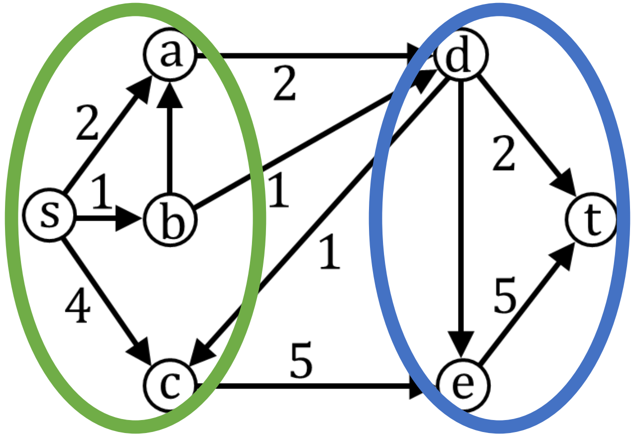
Theorem 1: Let G be a flow network, (A, B) be an $s - t$ cut, and f be an $s - t$ flow. Then, $|f| = \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{in}(A)} f(e)$.

Proof: Let $f^{out}(v) = \sum_{e \in \text{out}(v)} f(e)$ and $f^{in}(v) = \sum_{e \in \text{in}(v)} f(e)$.

Then, $|f| = \sum_{v \in A}(f^{out}(v) - f^{in}(v))$ (Only $\neq 0$ when $v = s$).

$\forall e = (u, v) \in E$:

1. ~~$u \in A, v \in A \rightarrow f(e)$ cancels out.~~
2. ~~$u \notin A, v \notin A \rightarrow f(e)$ does not appear at all.~~
3. $u \in A, v \notin A \rightarrow f(e)$ adds to the sum.
4. $u \notin A, v \in A$ **(edge enters A)**



Optimality

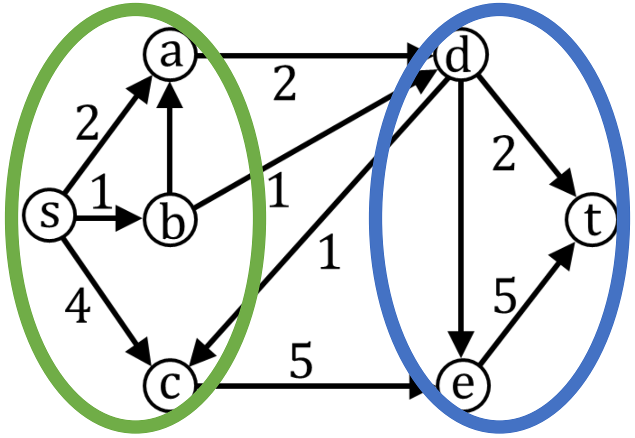
Theorem 1: Let G be a flow network, (A, B) be an $s - t$ cut, and f be an $s - t$ flow. Then, $|f| = \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{in}(A)} f(e)$.

Proof: Let $f^{out}(v) = \sum_{e \in \text{out}(v)} f(e)$ and $f^{in}(v) = \sum_{e \in \text{in}(v)} f(e)$.

Then, $|f| = \sum_{v \in A}(f^{out}(v) - f^{in}(v))$ (Only $\neq 0$ when $v = s$).

$\forall e = (u, v) \in E$:

1. ~~$u \in A, v \in A \rightarrow f(e)$ cancels out.~~
2. ~~$u \notin A, v \notin A \rightarrow f(e)$ does not appear at all.~~
3. $u \in A, v \notin A \rightarrow f(e)$ adds to the sum.
4. $u \notin A, v \in A \rightarrow f(e)$ subtracts from the sum.



Optimality

Theorem 1: Let G be a flow network, (A, B) be an $s-t$ cut, and f be an $s-t$ flow. Then, $|f| = \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{in}(A)} f(e)$.

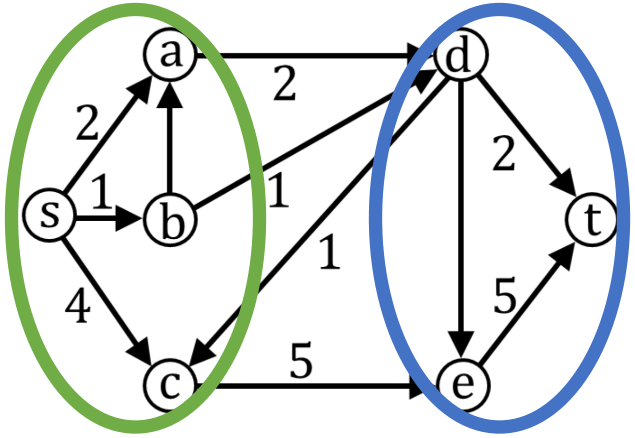
Proof: Let $f^{out}(v) = \sum_{e \in \text{out}(v)} f(e)$ and $f^{in}(v) = \sum_{e \in \text{in}(v)} f(e)$.

Then, $|f| = \sum_{v \in A}(f^{out}(v) - f^{in}(v))$ (Only $\neq 0$ when $v = s$).

$\forall e = (u, v) \in E$:

1. ~~$u \in A, v \in A \rightarrow f(e)$ cancels out.~~
2. ~~$u \notin A, v \notin A \rightarrow f(e)$ does not appear at all.~~
3. $u \in A, v \notin A \rightarrow f(e)$ adds to the sum.
4. $u \notin A, v \in A \rightarrow f(e)$ subtracts from the sum.

$$\Longrightarrow \sum_{v \in A}(f^{out}(v) - f^{in}(v)) = \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{in}(A)} f(e)$$



Optimality

Theorem 1: Let G be a flow network, (A, B) be an $s - t$ cut, and f be an $s - t$ flow.
Then, $|f| = \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{in}(A)} f(e)$.

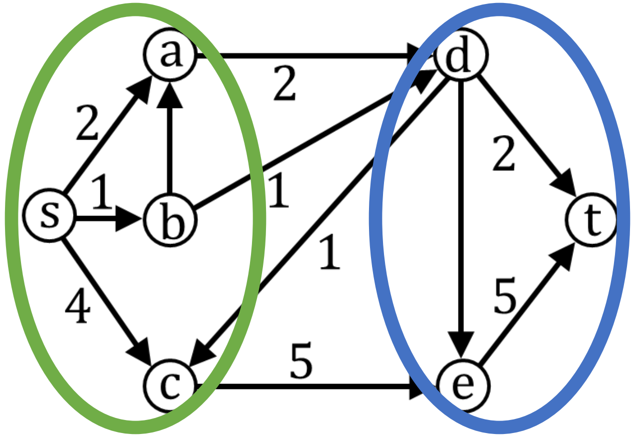
Proof: Let $f^{out}(v) = \sum_{e \in \text{out}(v)} f(e)$ and $f^{in}(v) = \sum_{e \in \text{in}(v)} f(e)$.

Then, $|f| = \sum_{v \in A} (f^{out}(v) - f^{in}(v))$ (Only $\neq 0$ when $v = s$).

$\forall e = (u, v) \in E$:

1. ~~$u \in A, v \in A \rightarrow f(e)$ cancels out.~~
2. ~~$u \notin A, v \notin A \rightarrow f(e)$ does not appear at all.~~
3. $u \in A, v \notin A \rightarrow f(e)$ adds to the sum.
4. $u \notin A, v \in A \rightarrow f(e)$ subtracts from the sum.

$$\Longrightarrow \sum_{v \in A} (f^{out}(v) - f^{in}(v))$$
$$= \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{in}(A)} f(e)$$
$$\Longrightarrow |f| = \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{in}(A)} f(e)$$



Optimality

This relates arbitrary $s - t$ flows to arbitrary $s - t$ cuts

<u>Theorem 1:</u> Let G be a flow network, (A, B) be an $s - t$ cut, and f be an $s - t$ flow. Then, $|f| = \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{in}(A)} f(e)$.

Optimality

This relates arbitrary $s - t$ flows to arbitrary $s - t$ cuts

Theorem 1: Let G be a flow network, (A, B) be an $s - t$ cut, and f be an $s - t$ flow. Then, $|f| = \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{in}(A)} f(e)$.

Corollary: Suppose G is a flow network, f is an $s - t$ flow on G, and (A, B) is an $s - t$ cut. Then, $|f| \leq c(A, B)$. (i.e. *every* flow is bounded by *any* $s - t$ cut)

Proof:

?

Optimality

<u>Theorem 1:</u> Let G be a flow network, (A, B) be an $s - t$ cut, and f be an $s - t$ flow. Then, $|f| = \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{in}(A)} f(e)$.

<u>Corollary:</u> Suppose G is a flow network, f is an $s - t$ flow on G, and (A, B) is an $s - t$ cut. Then, $|f| \leq c(A, B)$. (i.e. *every* flow is bounded by *any* $s - t$ cut)

<u>Proof:</u>

$$|f| = \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{in}(A)} f(e)$$

Optimality

Theorem 1: Let G be a flow network, (A, B) be an $s-t$ cut, and f be an $s-t$ flow. Then, $|f| = \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{in}(A)} f(e)$.

Corollary: Suppose G is a flow network, f is an $s-t$ flow on G, and (A, B) is an $s-t$ cut. Then, $|f| \leq c(A, B)$. (i.e. *every* flow is bounded by *any* $s-t$ cut)

Proof:

$$
\begin{aligned}
|f| &= \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{in}(A)} f(e) \\
&\leq \sum_{e \in \text{out}(A)} f(e)
\end{aligned}
$$

Optimality

Theorem 1: Let G be a flow network, (A, B) be an $s - t$ cut, and f be an $s - t$ flow. Then, $|f| = \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{in}(A)} f(e)$.

Corollary: Suppose G is a flow network, f is an $s - t$ flow on G, and (A, B) is an $s - t$ cut. Then, $|f| \leq c(A, B)$. (i.e. *every* flow is bounded by *any* $s - t$ cut)

Proof:

$$
\begin{aligned}
|f| &= \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{in}(A)} f(e) \\
&\leq \sum_{e \in \text{out}(A)} f(e) \\
&\leq \sum_{e \in \text{out}(A)} c_e = c(A, B)
\end{aligned}
$$

Optimality

Theorem 1: Let G be a flow network, (A, B) be an $s - t$ cut, and f be an $s - t$ flow. Then, $|f| = \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{in}(A)} f(e)$.

Corollary: Suppose G is a flow network, f is an $s - t$ flow on G, and (A, B) is an $s - t$ cut. Then, $|f| \leq c(A, B)$. (i.e. *every* flow is bounded by *any* $s - t$ cut)

Proof:

$$|f| = \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{in}(A)} f(e)$$
$$\leq \sum_{e \in \text{out}(A)} f(e)$$
$$\leq \sum_{e \in \text{out}(A)} c_e = c(A, B)$$

If we find some flow f and some cut (A, B) such that $|f| = c(A, B)$, then ?

Optimality

<u>Theorem 1:</u> Let G be a flow network, (A, B) be an $s-t$ cut, and f be an $s-t$ flow. Then, $|f| = \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{in}(A)} f(e)$.

<u>Corollary:</u> Suppose G is a flow network, f is an $s-t$ flow on G, and (A, B) is an $s-t$ cut. Then, $|f| \leq c(A, B)$. (i.e. *every* flow is bounded by *any* $s-t$ cut)

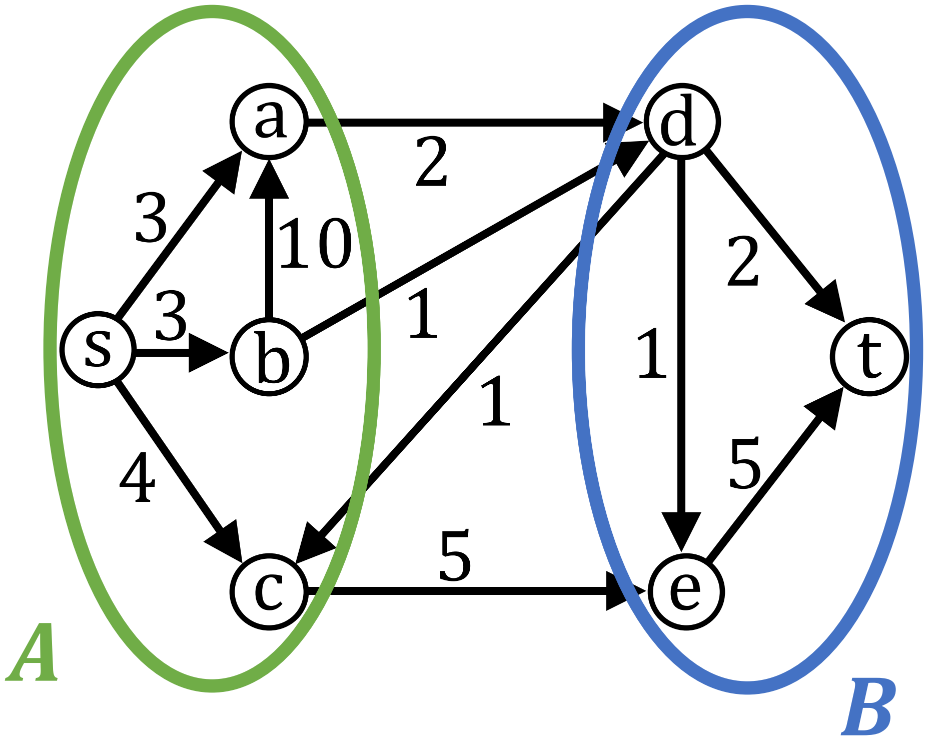
<u>Proof:</u>

$$
\begin{aligned}
|f| &= \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{in}(A)} f(e) \\
&\leq \sum_{e \in \text{out}(A)} f(e) \\
&\leq \sum_{e \in \text{out}(A)} c_e = c(A, B)
\end{aligned}
$$

If we find some flow f and some cut (A, B) such that $|f| = c(A, B)$, then f is a maximum flow.

Optimality

<u>Definitions:</u> Suppose G is a flow network and nodes in G are divided into two sets, $\boldsymbol{A}$ and $\boldsymbol{B}$, such that $s \in A$ and $t \in B$. We call (A, B) an $s - t$ *cut*. The *capacity* of the cut, $c(A, B)$, is the sum of capacities of all edges out of A.



$$c(A, B) = 8$$

Game Plan:

1. Show the value of every flow is ≤ capacity of every cut.
2. Given a flow where there are no $s - t$ paths left in the residual graph, there is a specific cut whose capacity = flow value.

Optimality

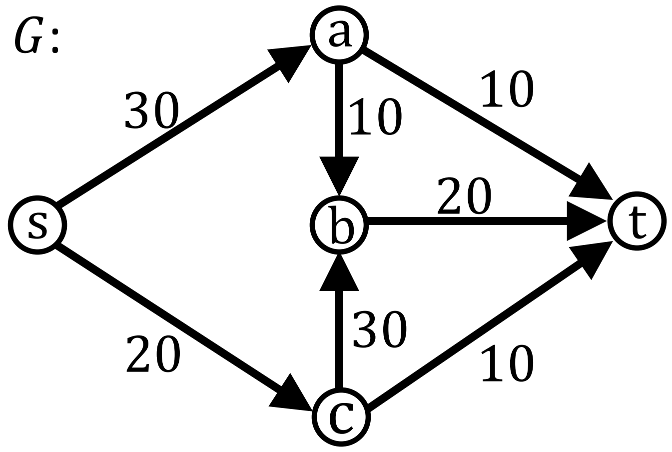
Theorem 2: if f is an $s-t$ flow such that no $s-t$ path exists in residual graph G_f, then there is an $s-t$ cut (A,B) in $G=(V,E)$ for which $|f|=c(A,B)$.

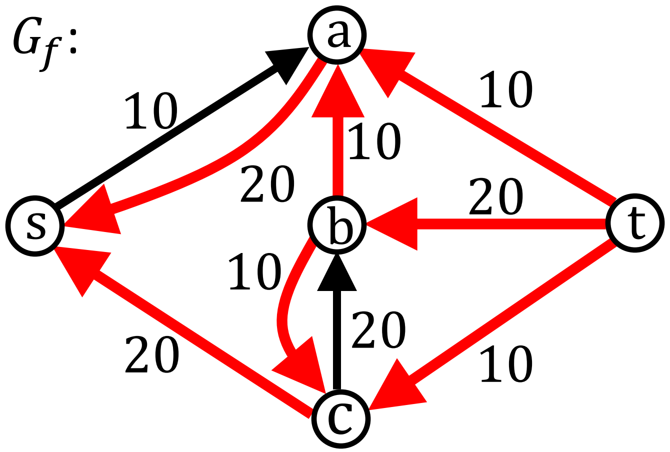
Proof:

Optimality

Theorem 2: if f is an $s-t$ flow such that no $s-t$ path exists in residual graph G_f, then there is an $s-t$ cut (A, B) in $G=(V, E)$ for which $|f|=c(A, B)$.

Proof:

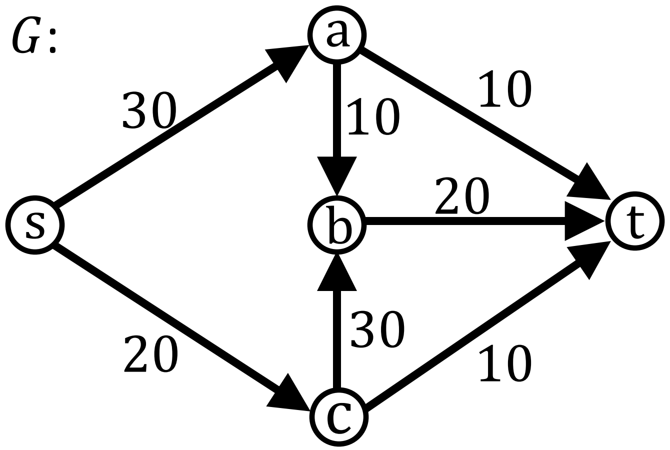


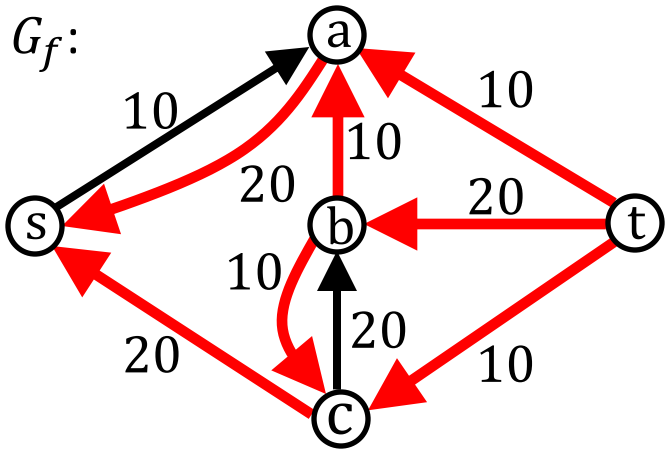


Optimality

<u>Theorem 2:</u> if f is an $s-t$ flow such that no $s-t$ path exists in residual graph G_f, then there is an $s-t$ cut (A, B) in $G=(V,E)$ for which $|f| = c(A,B)$.

<u>Proof:</u> Let $A = \{v \in V : \exists\ s-v \text{ path in } G_f\}$ and $B = V \setminus A$.

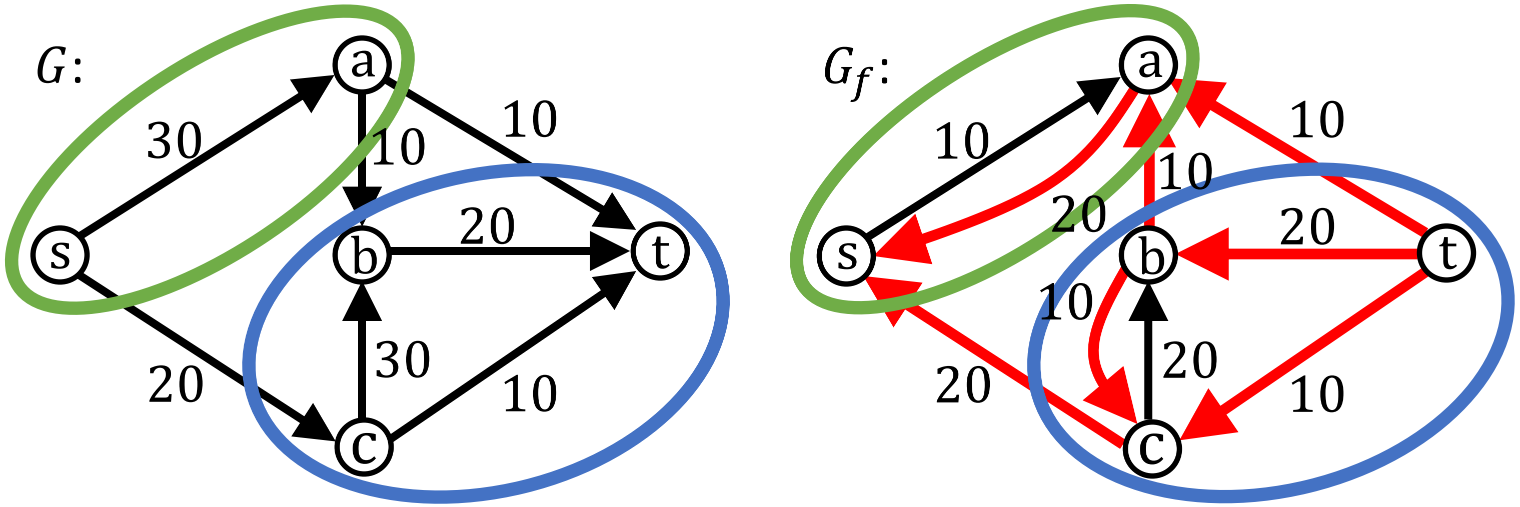




Optimality

<u>Theorem 2:</u> if f is an $s-t$ flow such that no $s-t$ path exists in residual graph G_f, then there is an $s-t$ cut (A,B) in $G=(V,E)$ for which $|f|=c(A,B)$.

<u>Proof:</u> Let $A=\{v \in V : \exists\ s-v \text{ path in } G_f\}$ and $B=V \backslash A$.

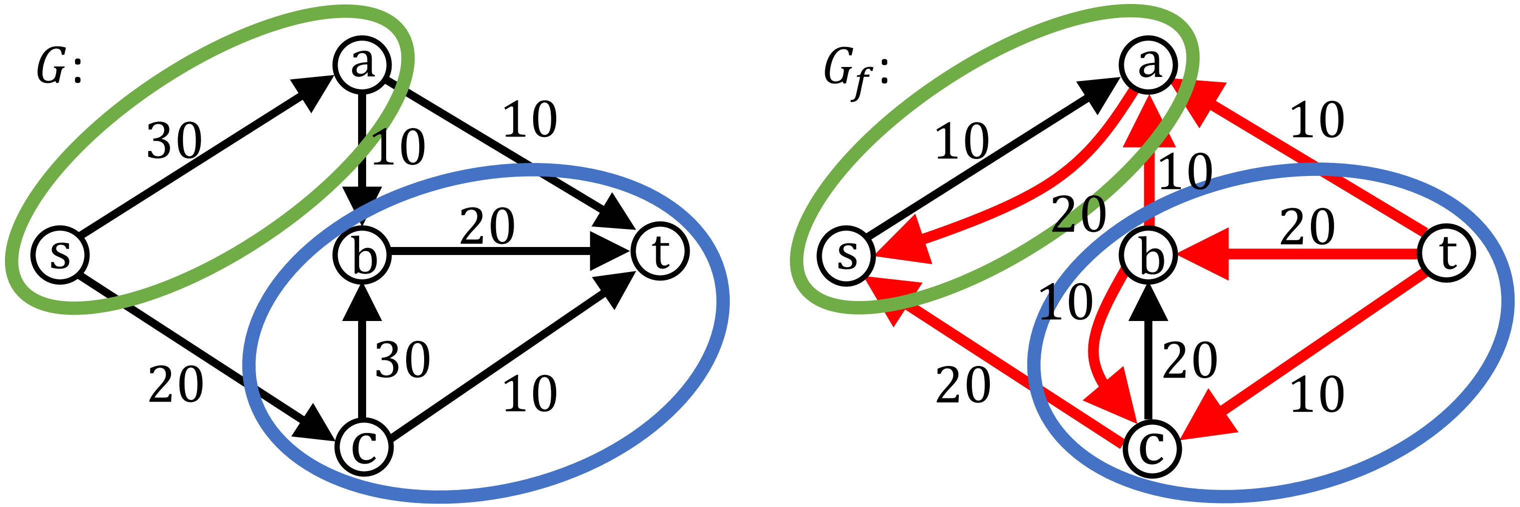


Optimality

Theorem 2: if f is an $s-t$ flow such that no $s-t$ path exists in residual graph G_f, then there is an $s-t$ cut (A,B) in $G=(V,E)$ for which $|f|=c(A,B)$.

Proof: Let $A=\{v \in V: \exists\ s-v \text{ path in } G_f\}$ and $B=V \backslash A$.

(A,B) is an $s-t$ cut (because ?)

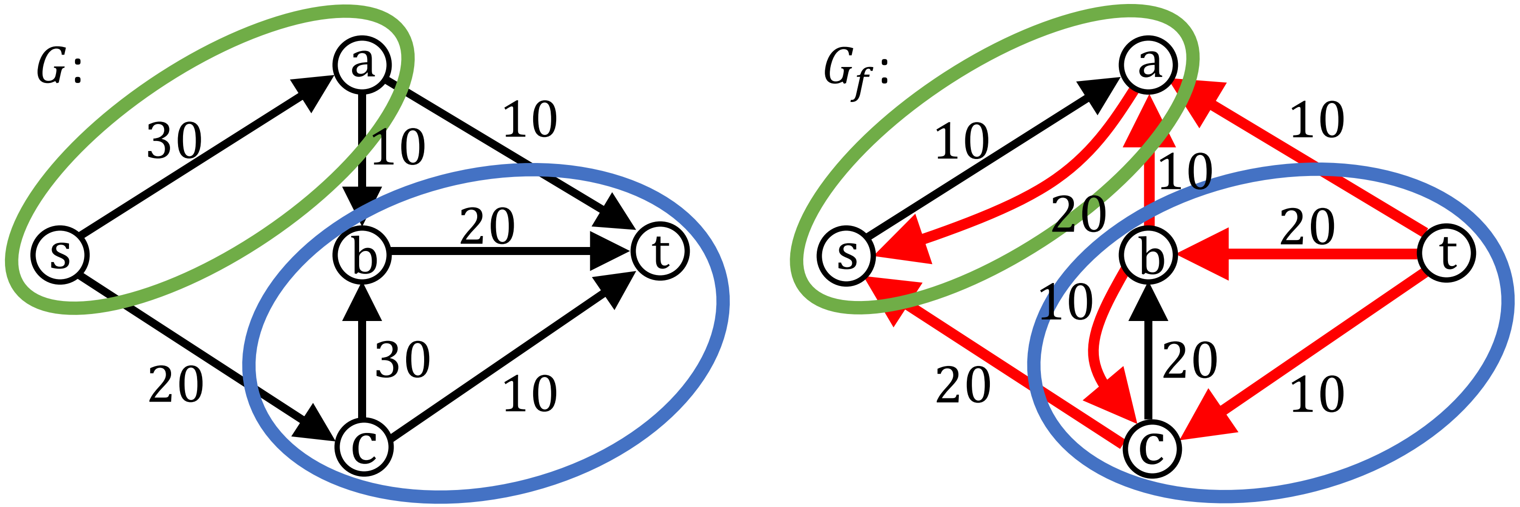


Optimality

Theorem 2: if f is an $s-t$ flow such that no $s-t$ path exists in residual graph G_f, then there is an $s-t$ cut (A,B) in $G=(V,E)$ for which $|f|=c(A,B)$.

Proof: Let $A=\{v\in V: \exists\ s-v \text{ path in } G_f\}$ and $B=V\setminus A$.

(A,B) is an $s-t$ cut (because it partitions V, $s\in A$, and $t\in B$)



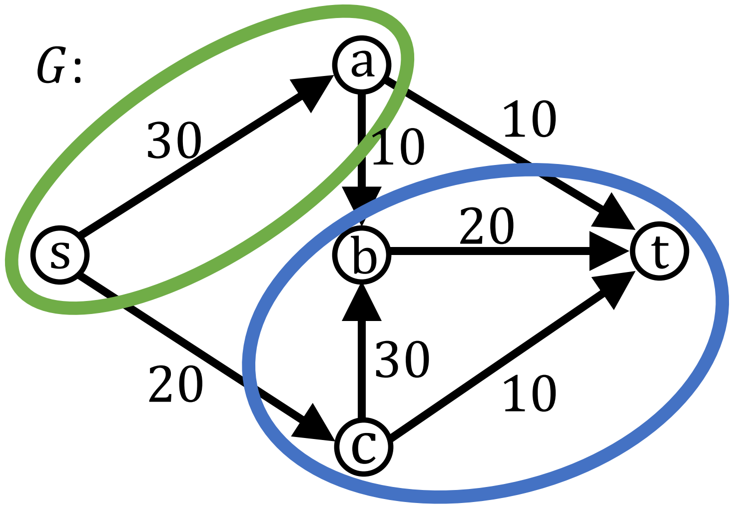
Optimality

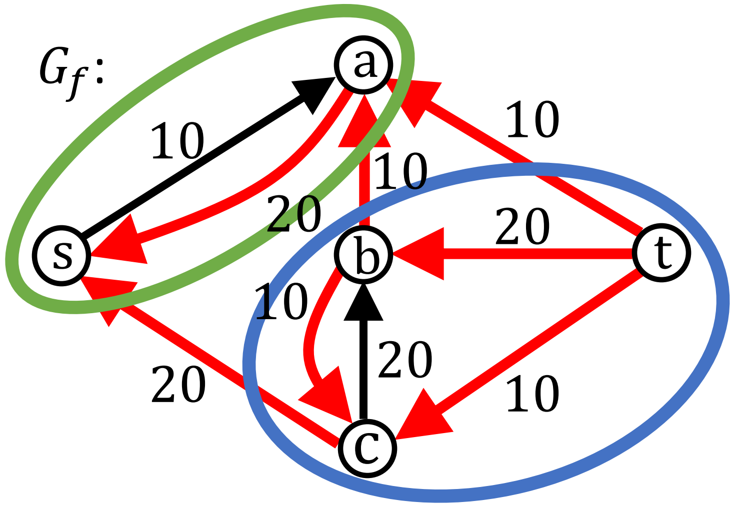
Theorem 2: if f is an $s-t$ flow such that no $s-t$ path exists in residual graph G_f, then there is an $s-t$ cut (A,B) in $G=(V,E)$ for which $|f|=c(A,B)$.

Proof: Let $A=\{v\in V: \exists\ s-v \text{ path in } G_f\}$ and $B=V\backslash A$.

(A,B) is an $s-t$ cut (because it partitions V, $s\in A$, and $t\in B$)

Need to compare flow across cut to capacity of cut.





Optimality

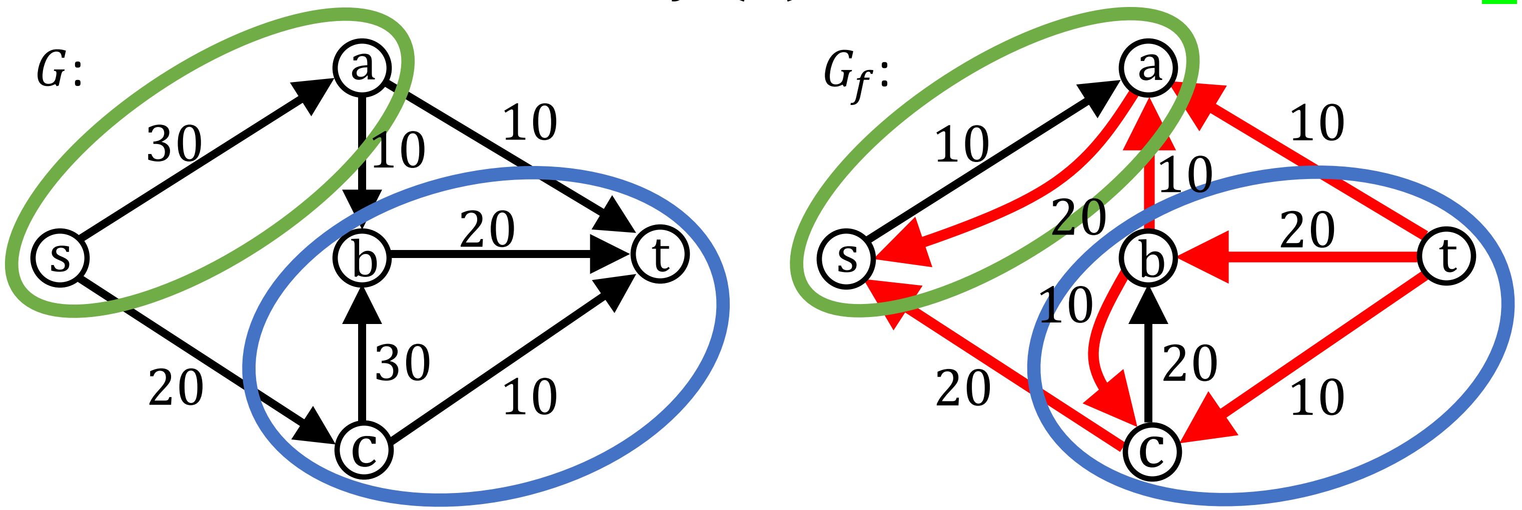
Theorem 2: if f is an $s-t$ flow such that no $s-t$ path exists in residual graph G_f, then there is an $s-t$ cut (A,B) in $G=(V,E)$ for which $|f|=c(A,B)$.

Proof: Let $A=\{v\in V: \exists\ s-v \text{ path in } G_f\}$ and $B=V\setminus A$.

(A,B) is an $s-t$ cut (because it partitions V, $s\in A$, and $t\in B$)

Let $e=(u,v)\in E$ (directed edge) such that $u\in A$ and $v\in B$.

What can we say about $f(e)$ related to its capacity?



Optimality

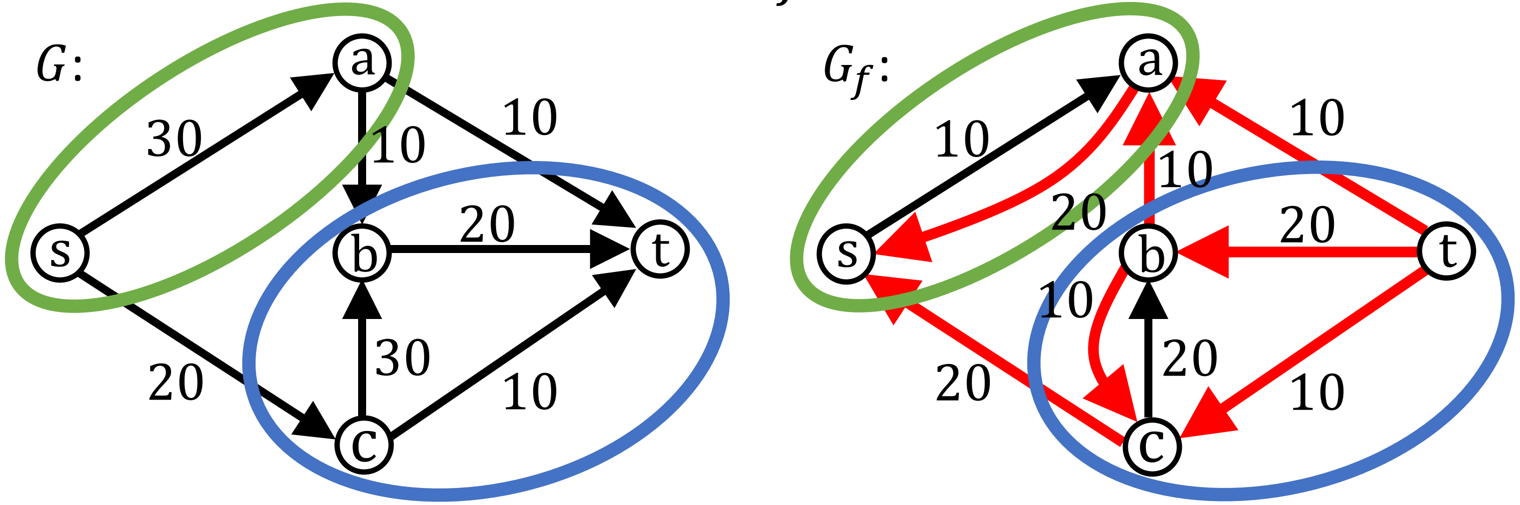
<u>Theorem 2:</u> if f is an $s-t$ flow such that no $s-t$ path exists in residual graph G_f, then there is an $s-t$ cut (A,B) in $G=(V,E)$ for which $|f|=c(A,B)$.

<u>Proof:</u> Let $A=\{v \in V: \exists\ s-v \text{ path in } G_f\}$ and $B=V\backslash A$.

(A,B) is an $s-t$ cut (because it partitions V, $s \in A$, and $t \in B$)

Let $e=(u,v) \in E$ (directed edge) such that $u \in A$ and $v \in B$.

$f(e)=c_e$ (since $(u,v) \notin G_f$, otherwise v would be in A)



Optimality

Theorem 2: if f is an $s - t$ flow such that no $s - t$ path exists in residual graph G_f, then there is an $s - t$ cut (A, B) in $G = (V, E)$ for which $|f| = c(A, B)$.

Proof: Let $A = \{v \in V : \exists\ s - v$ path in $G_f\}$ and $B = V \setminus A$.

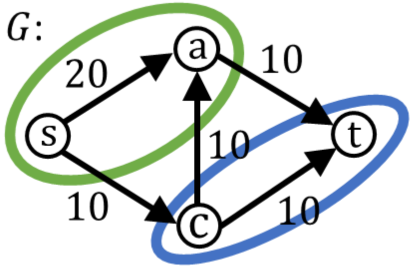
(A, B) is an $s - t$ cut (because it partitions V, $s \in A$, and $t \in B$)

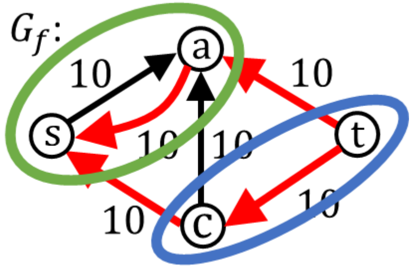
Let $e = (u, v) \in E$ (directed edge) such that $u \in A$ and $v \in B$.

$f(e) = c_e$ (since $(u, v) \notin G_f$, otherwise v would be in A)

Let $e' = (u', v') \in E$ (directed edge) such that $u' \in B$ and $v' \in A$.

What can we say about $f(e')$?





Optimality

Theorem 2: if f is an $s - t$ flow such that no $s - t$ path exists in residual graph G_f, then there is an $s - t$ cut (A, B) in $G = (V, E)$ for which $|f| = c(A, B)$.

Proof: Let $A = \{v \in V: \exists\ s - v \text{ path in } G_f\}$ and $B = V \backslash A$.

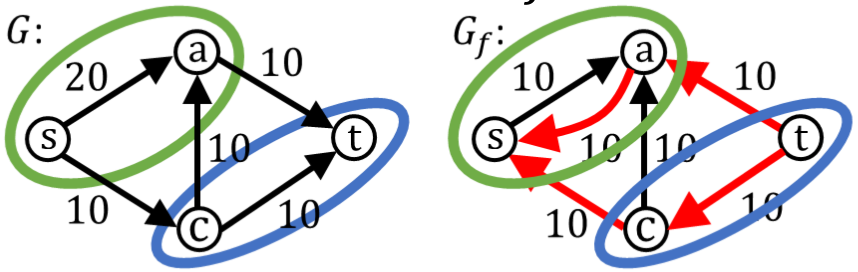
(A, B) is an $s - t$ cut (because it partitions V, $s \in A$, and $t \in B$)

Let $e = (u, v) \in E$ (directed edge) such that $u \in A$ and $v \in B$.

$f(e) = c_e$ (since $(u, v) \notin G_f$, otherwise v would be in A)

Let $e' = (u', v') \in E$ (directed edge) such that $u' \in B$ and $v' \in A$.

$f(e') = 0$ (since $(v', u') \notin G_f$, otherwise u' would be in A)



Optimality

Theorem 2: if f is an $s - t$ flow such that no $s - t$ path exists in residual graph G_f, then there is an $s - t$ cut (A, B) in $G = (V, E)$ for which $|f| = c(A, B)$.

Proof: Let $A = \{v \in V: \exists\ s - v$ path in $G_f\}$ and $B = V \setminus A$.

(A, B) is an $s - t$ cut (because it partitions V, $s \in A$, and $t \in B$)

Let $e = (u, v) \in E$ (directed edge) such that $u \in A$ and $v \in B$.

$f(e) = c_e$ (since $(u, v) \notin G_f$, otherwise v would be in A)

Let $e' = (u', v') \in E$ (directed edge) such that $u' \in B$ and $v' \in A$.

$f(e') = 0$ (since $(v', u') \notin G_f$, otherwise u' would be in A)

Therefore, $|f| = \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{in}(A)} f(e)$ (By Theorem 1)

Optimality

Theorem 2: if f is an $s-t$ flow such that no $s-t$ path exists in residual graph G_f, then there is an $s-t$ cut (A, B) in $G=(V, E)$ for which $|f|=c(A, B)$.

Proof: Let $A=\{v \in V: \exists\, s-v$ path in $G_f\}$ and $B=V \backslash A$.

(A, B) is an $s-t$ cut (because it partitions V, $s \in A$, and $t \in B$)

Let $e=(u, v) \in E$ (directed edge) such that $u \in A$ and $v \in B$.

$f(e)=c_e$ (since $(u, v) \notin G_f$, otherwise v would be in A)

Let $e'=(u', v') \in E$ (directed edge) such that $u' \in B$ and $v' \in A$.

$f(e')=0$ (since $(v', u') \notin G_f$, otherwise u' would be in A)

Therefore, $|f|=\sum_{e \in \text{out}(A)} f(e)-\sum_{e \in \text{in}(A)} f(e)$ (By Theorem 1)

$=\sum_{e \in \text{out}(A)} c_e-0=c(A, B)$

Optimality

Theorem: The flow returned by the Ford-Fulkerson algorithm is a maximum flow.

Proof:

??

Corollary: Suppose G is a flow network, f is an $s-t$ flow on G, and (A, B) is an $s-t$ cut. Then, $|f| \leq c(A, B)$. (i.e. *every* flow is bounded by *any* $s-t$ cut)

Theorem 2: if f is an $s-t$ flow such that no $s-t$ path exists in residual graph G_f, then there is an $s-t$ cut (A, B) in $G=(V, E)$ for which $|f| = c(A, B)$.

Optimality

Theorem: The flow returned by the Ford-Fulkerson algorithm is a maximum flow.

Proof: Since the Ford-Fulkerson algorithm finishes when no $s - t$ paths remain in G_f, Theorem 2 says there must be an $s - t$ cut such that the value of flow found equals the capacity of the cut.

Corollary: Suppose G is a flow network, f is an $s - t$ flow on G, and (A, B) is an $s - t$ cut. Then, $|f| \leq c(A, B)$. (i.e. *every* flow is bounded by *any* $s - t$ cut)

Theorem 2: if f is an $s - t$ flow such that no $s - t$ path exists in residual graph G_f, then there is an $s - t$ cut (A, B) in $G = (V, E)$ for which $|f| = c(A, B)$.

Optimality

<u>Theorem:</u> The flow returned by the Ford-Fulkerson algorithm is a maximum flow.

<u>Proof:</u> Since the Ford-Fulkerson algorithm finishes when no $s - t$ paths remain in G_f, Theorem 2 says there must be an $s - t$ cut such that the value of flow found equals the capacity of the cut.

By the Corollary, there cannot be a flow with a larger value.

<u>Corollary:</u> Suppose G is a flow network, f is an $s - t$ flow on G, and (A, B) is an $s - t$ cut. Then, $|f| \leq c(A, B)$. (i.e. *every* flow is bounded by *any* $s - t$ cut)

<u>Theorem 2:</u> if f is an $s - t$ flow such that no $s - t$ path exists in residual graph G_f, then there is an $s - t$ cut (A, B) in $G = (V, E)$ for which $|f| = c(A, B)$.

Optimality

<u>Theorem:</u> The flow returned by the Ford-Fulkerson algorithm is a maximum flow.

<u>Proof:</u> Since the Ford-Fulkerson algorithm finishes when no $s-t$ paths remain in G_f, Theorem 2 says there must be an $s-t$ cut such that the value of flow found equals the capacity of the cut.

By the Corollary, there cannot be a flow with a larger value.

Therefore, the flow found by the Ford-Fulkerson algorithm is the maximum flow.

<u>Corollary:</u> Suppose G is a flow network, f is an $s-t$ flow on G, and (A, B) is an $s-t$ cut. Then, $|f| \leq c(A, B)$. (i.e. *every* flow is bounded by *any* $s-t$ cut)

<u>Theorem 2:</u> if f is an $s-t$ flow such that no $s-t$ path exists in residual graph G_f, then there is an $s-t$ cut (A, B) in $G = (V, E)$ for which $|f| = c(A, B)$.